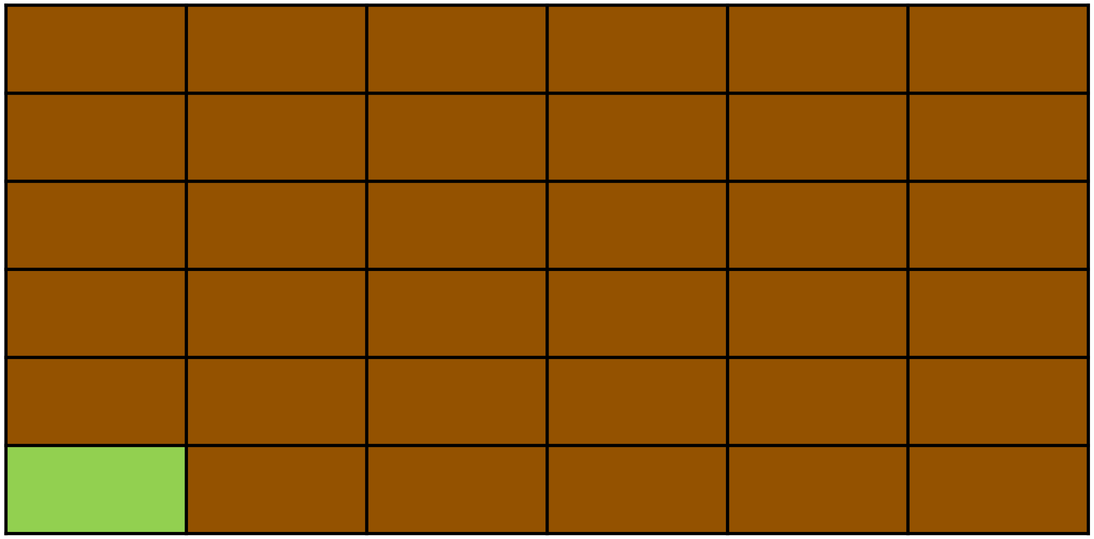
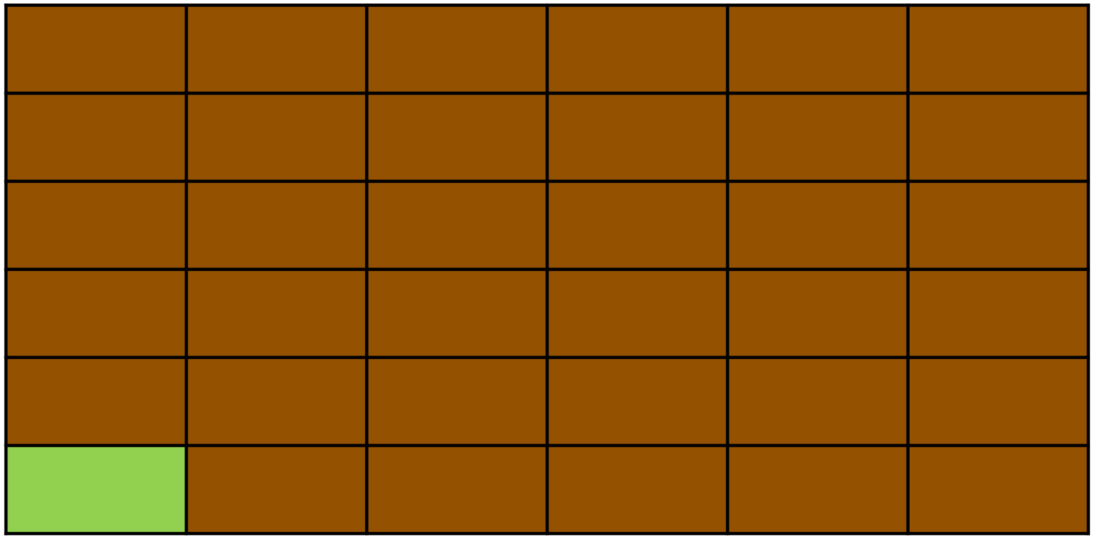
Chomp

Here is a chocolate bar



Chomp

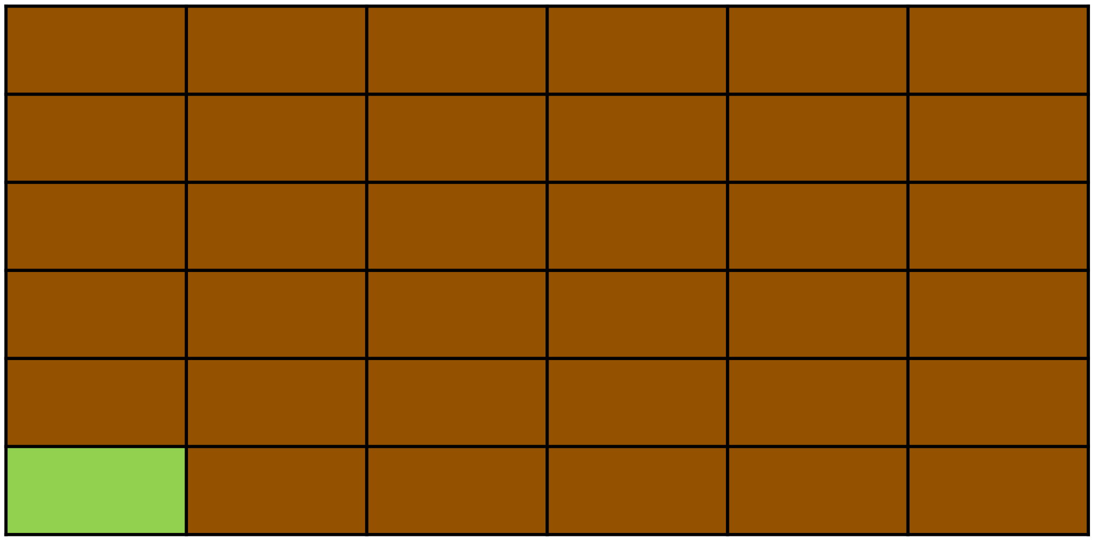
Here is a chocolate bar



Unfortunately, one piece is poisoned

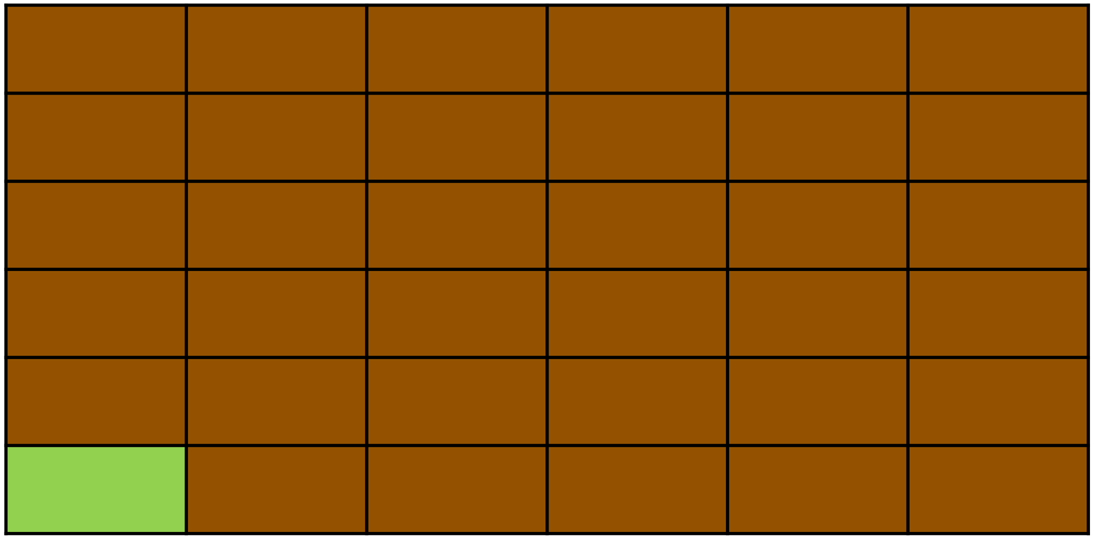
Chomp

Pick a piece – take everything above and to the right



Chomp

Pick a piece – take everything above and to the right

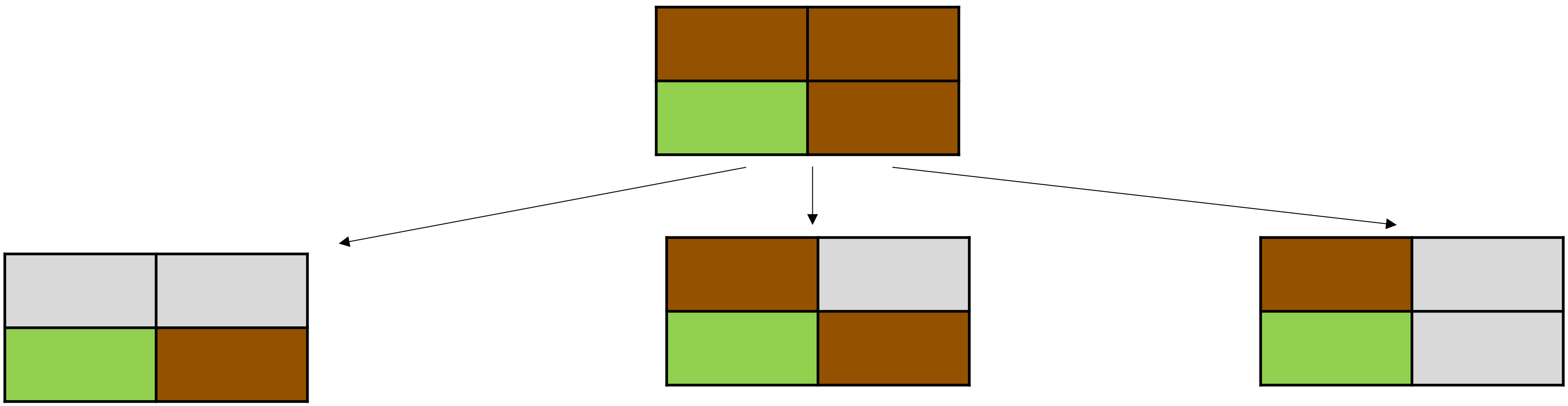


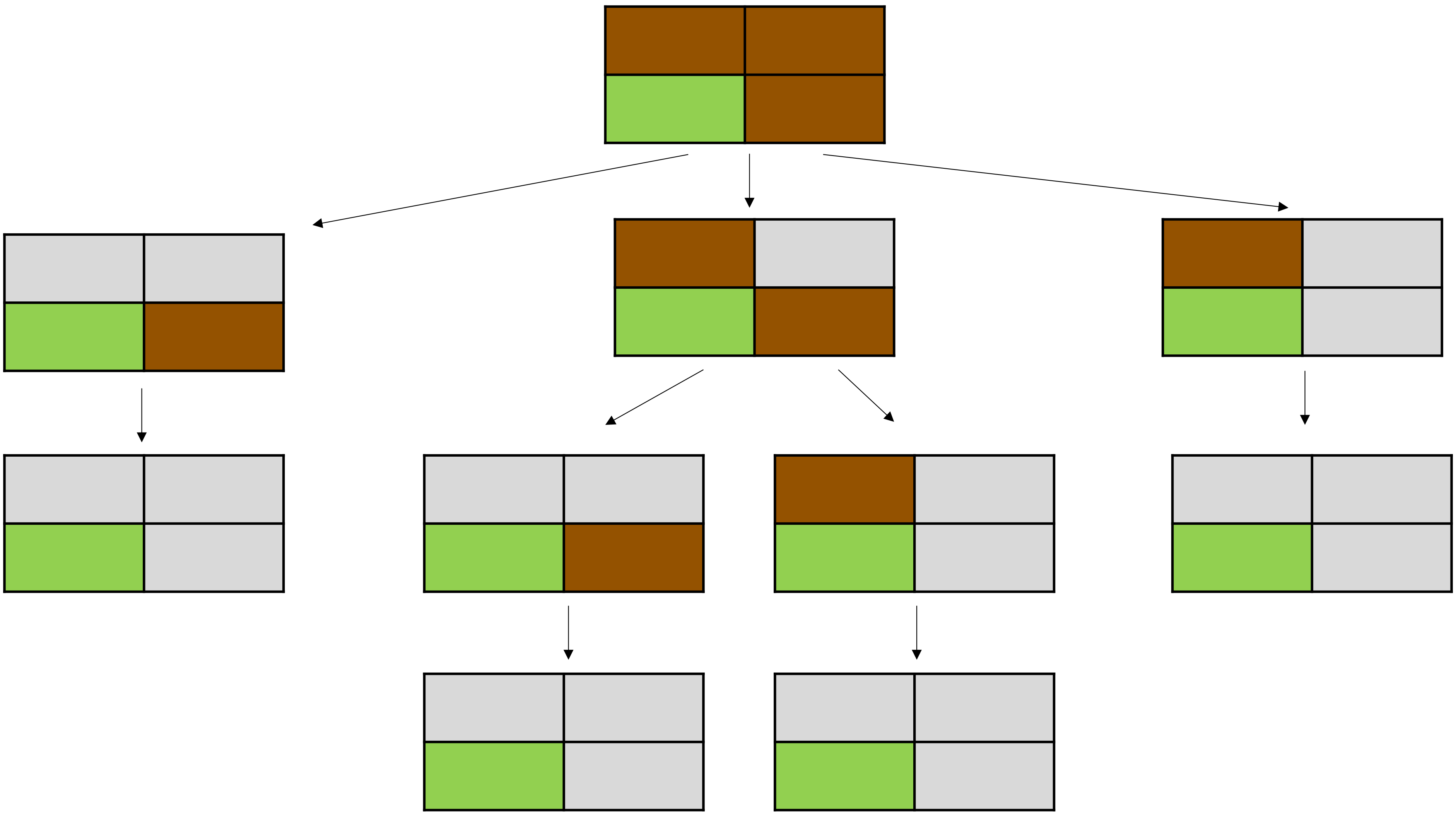
Try not to eat the poisoned piece

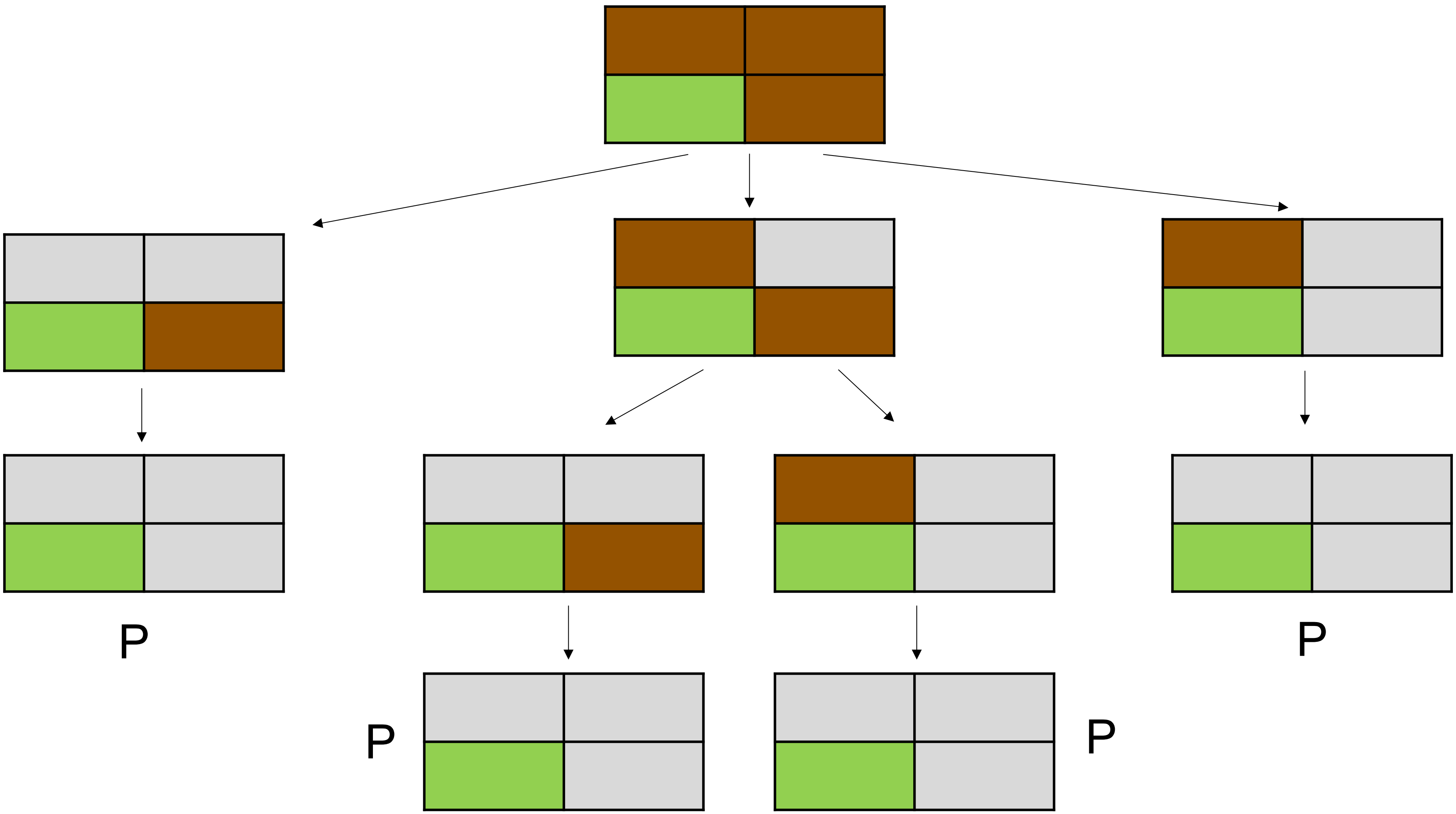
Over to Neel

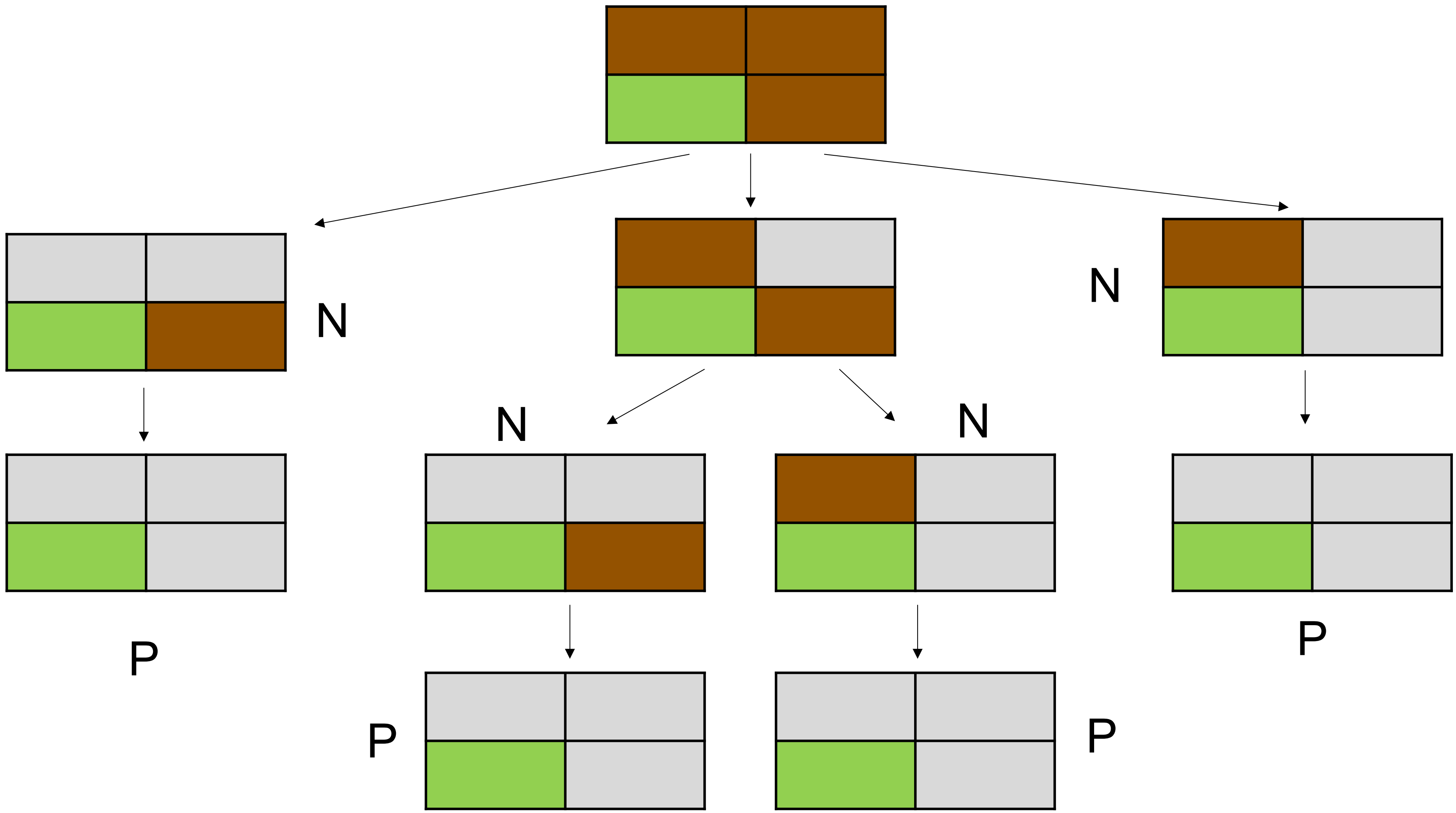
Tiny Chomp

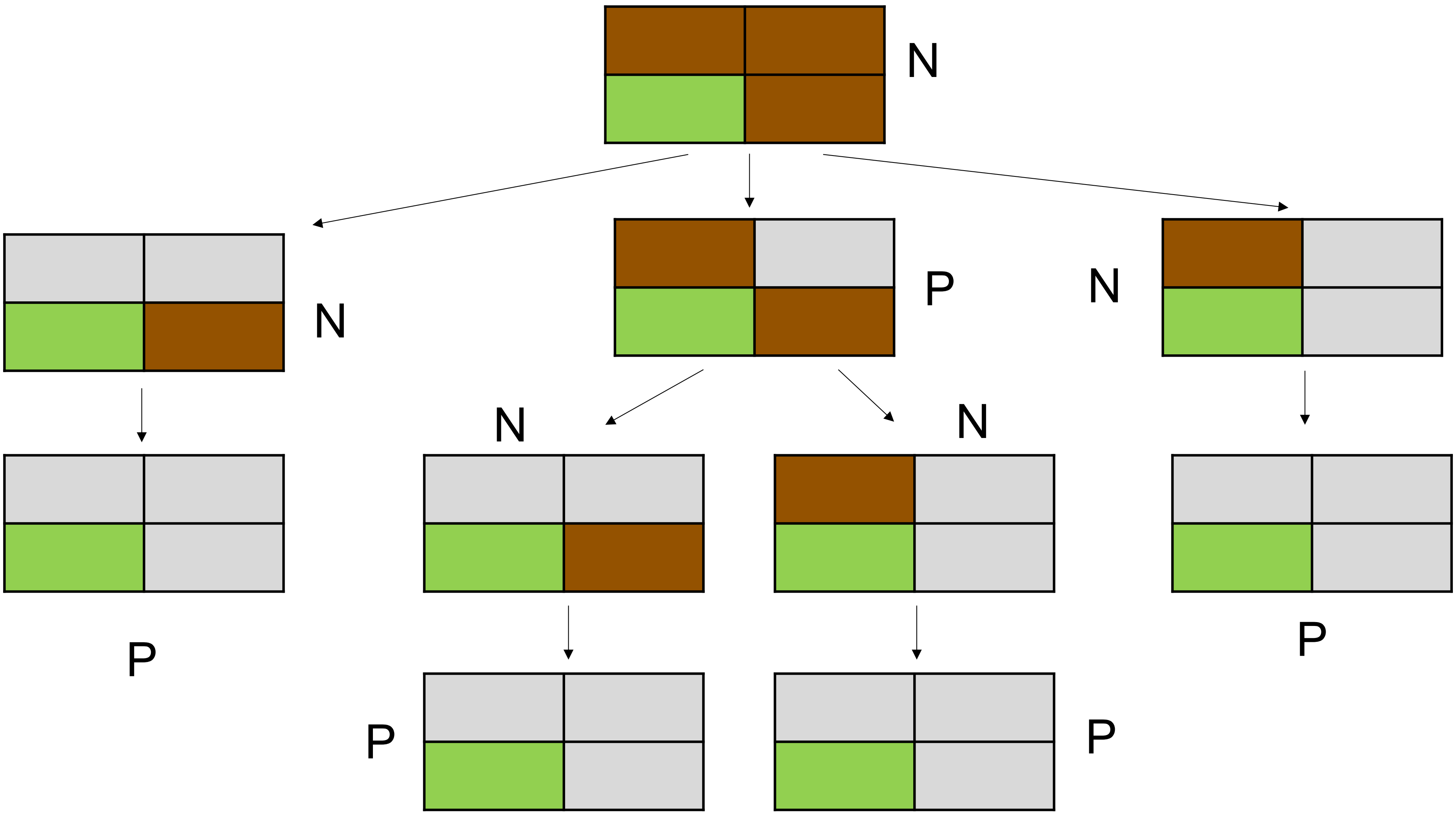
Who feels ill tomorrow?






P
P
P
P


N
N
N
N
P
P
P
P


N
N
P
N
N
N
P
P
P
P

Combinatorial Games

Game board and rules

No hidden information

Two player

No Chance

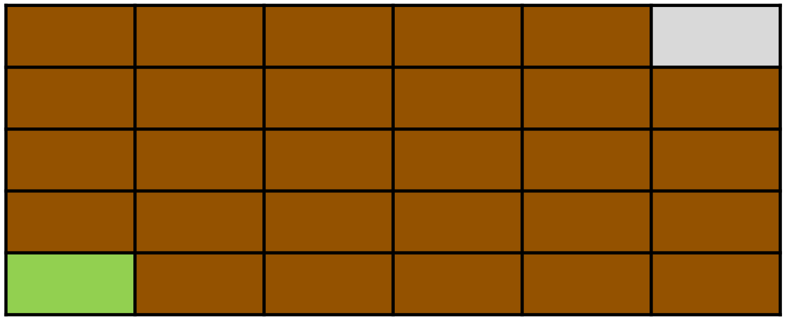
Turn-based

Terminates in finite steps

Fundamental Theorem

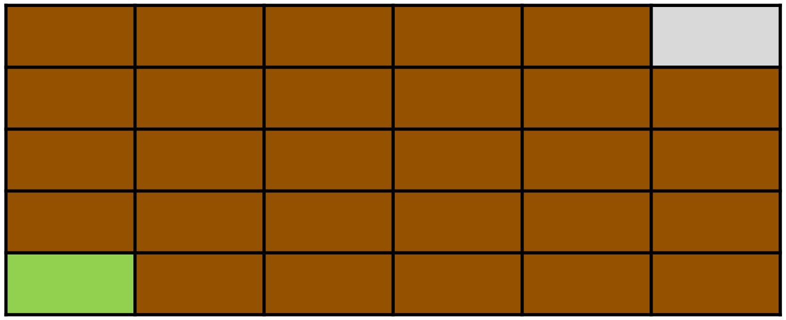
Either the first player or the second can force a win – not both

First player wins chomp



First player wins chomp

Say player 1 takes the top right square



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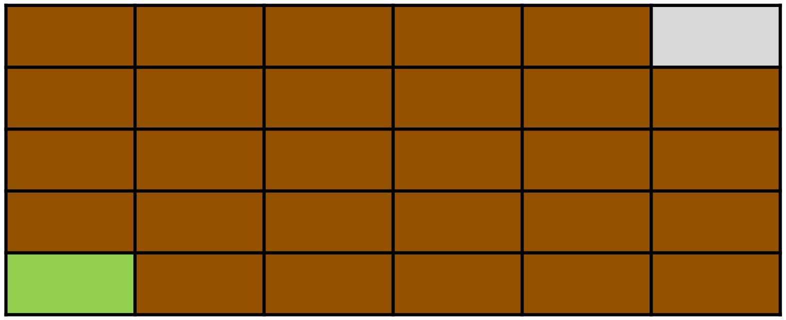
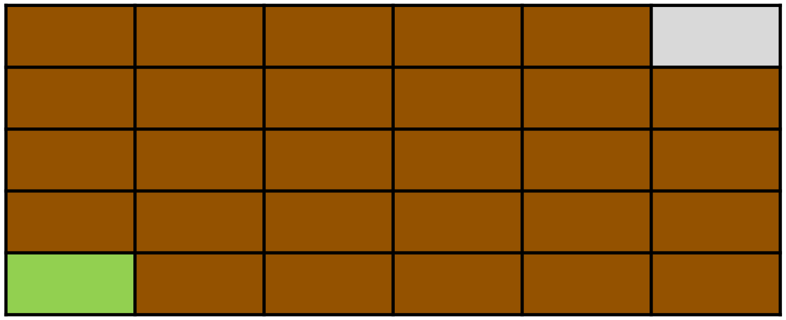


Either this is a winning first move or it is not

First player wins chomp

Say player 1 takes the top right square

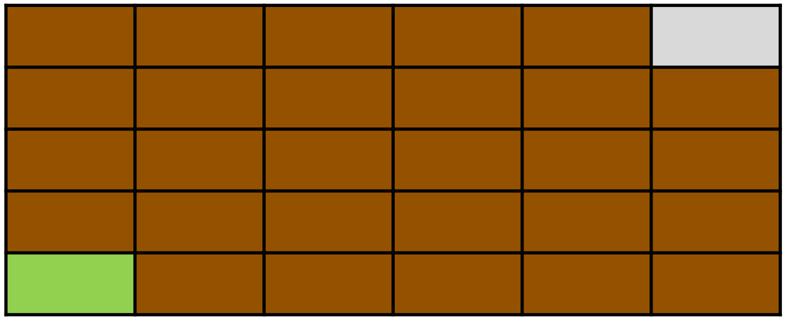


Either this is a winning first move or it is not

If losing move, 2nd player can respond with a winning move

First player wins chomp

Say player 1 takes the top right square



Either this is a winning first move or it is not

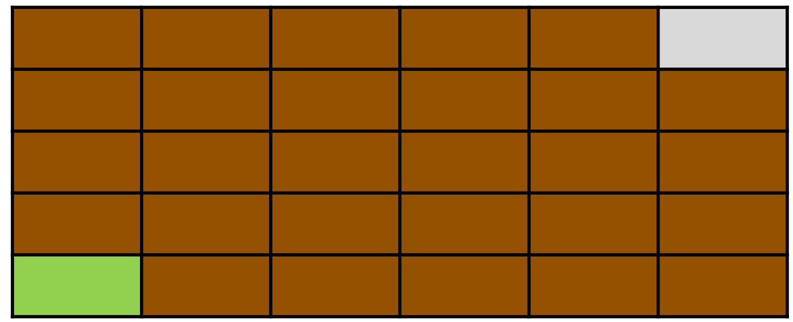
If losing move, 2nd player can respond with a winning move

But, no matter where the 2nd player chomps, player 1 had access to it

First player wins chomp

either by taking top right or some other piece

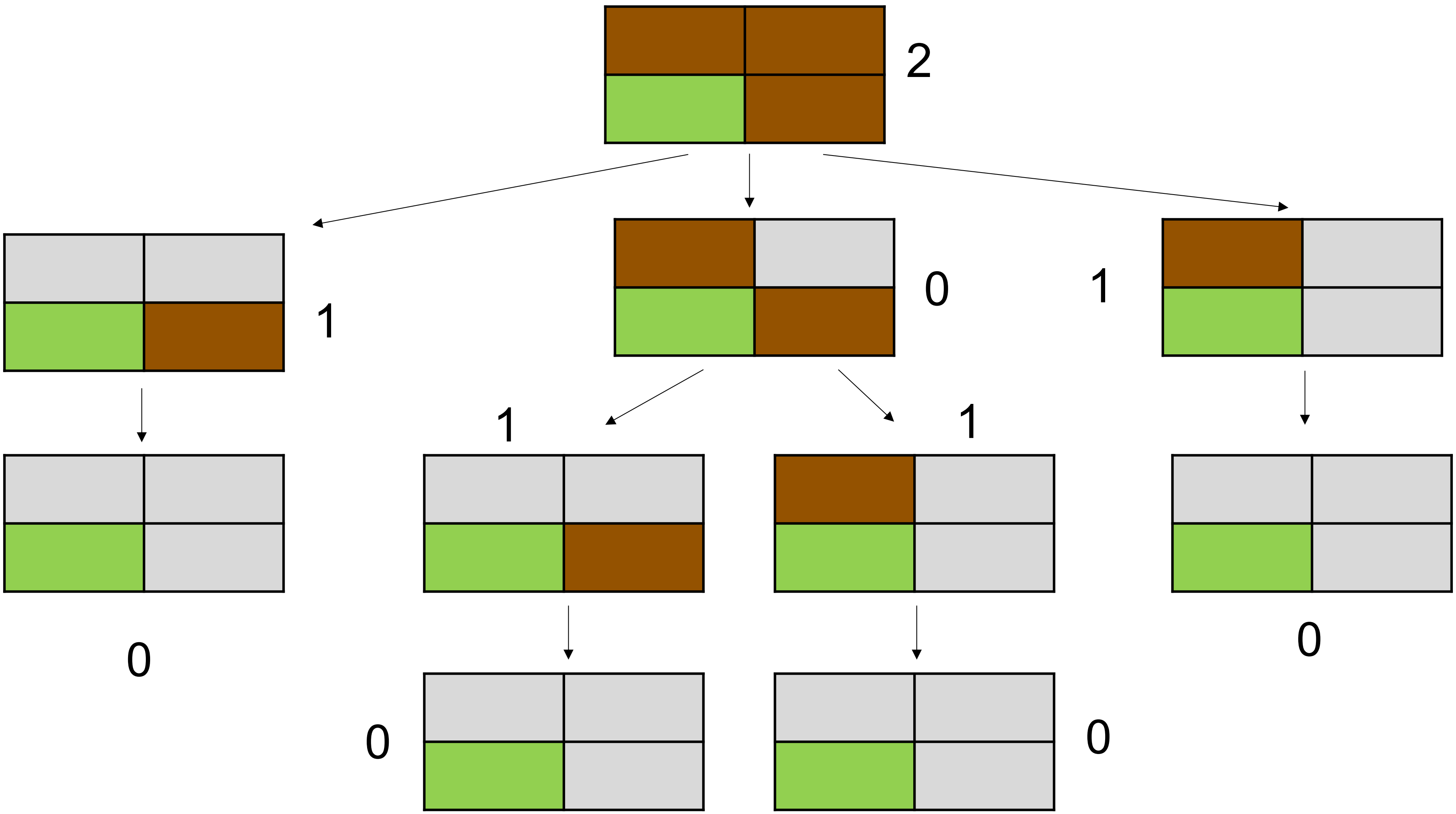
Say player 1 takes the top right square



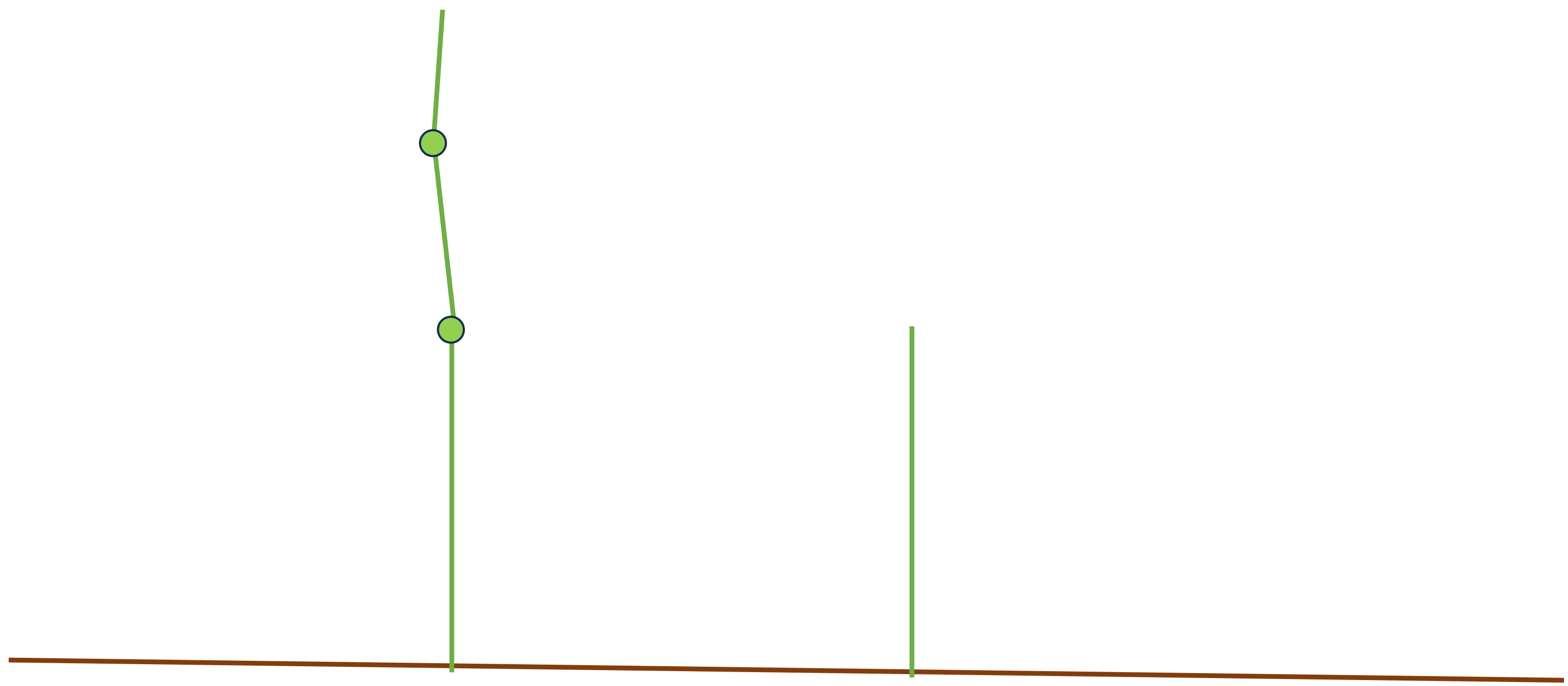
Either this is a winning first move or it is not

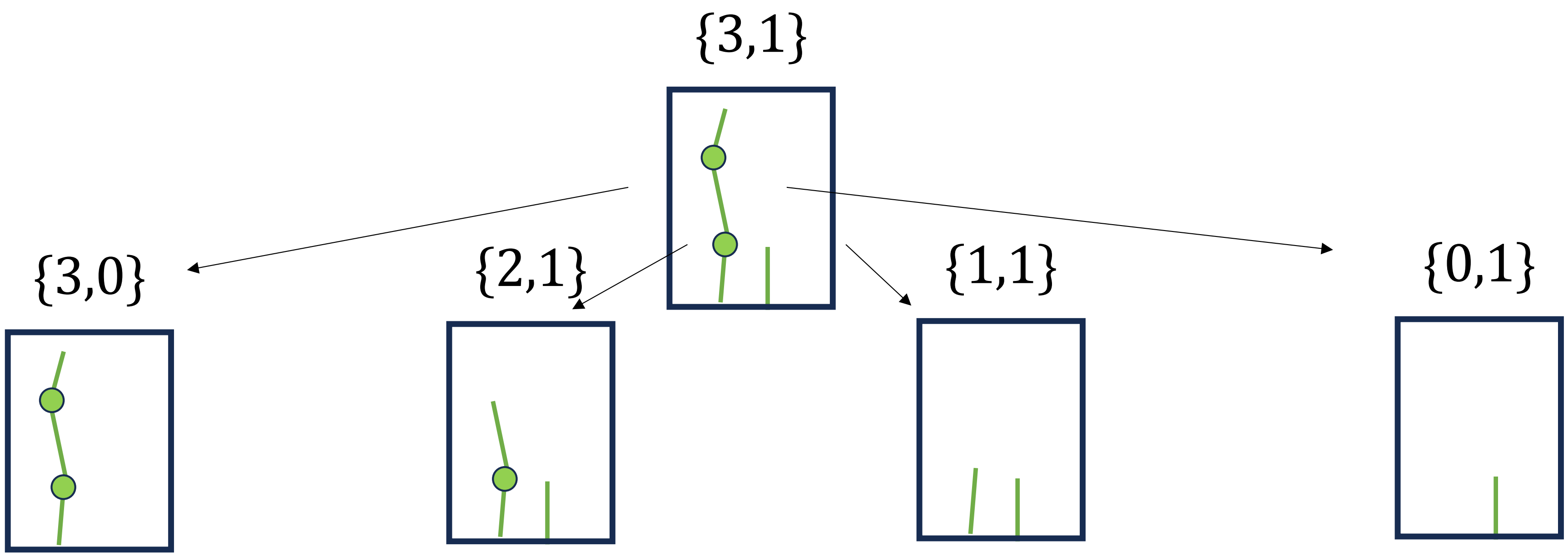
If losing move, 2nd player can respond with a winning move

But, no matter where the 2nd player chomps, player 1 had access to it

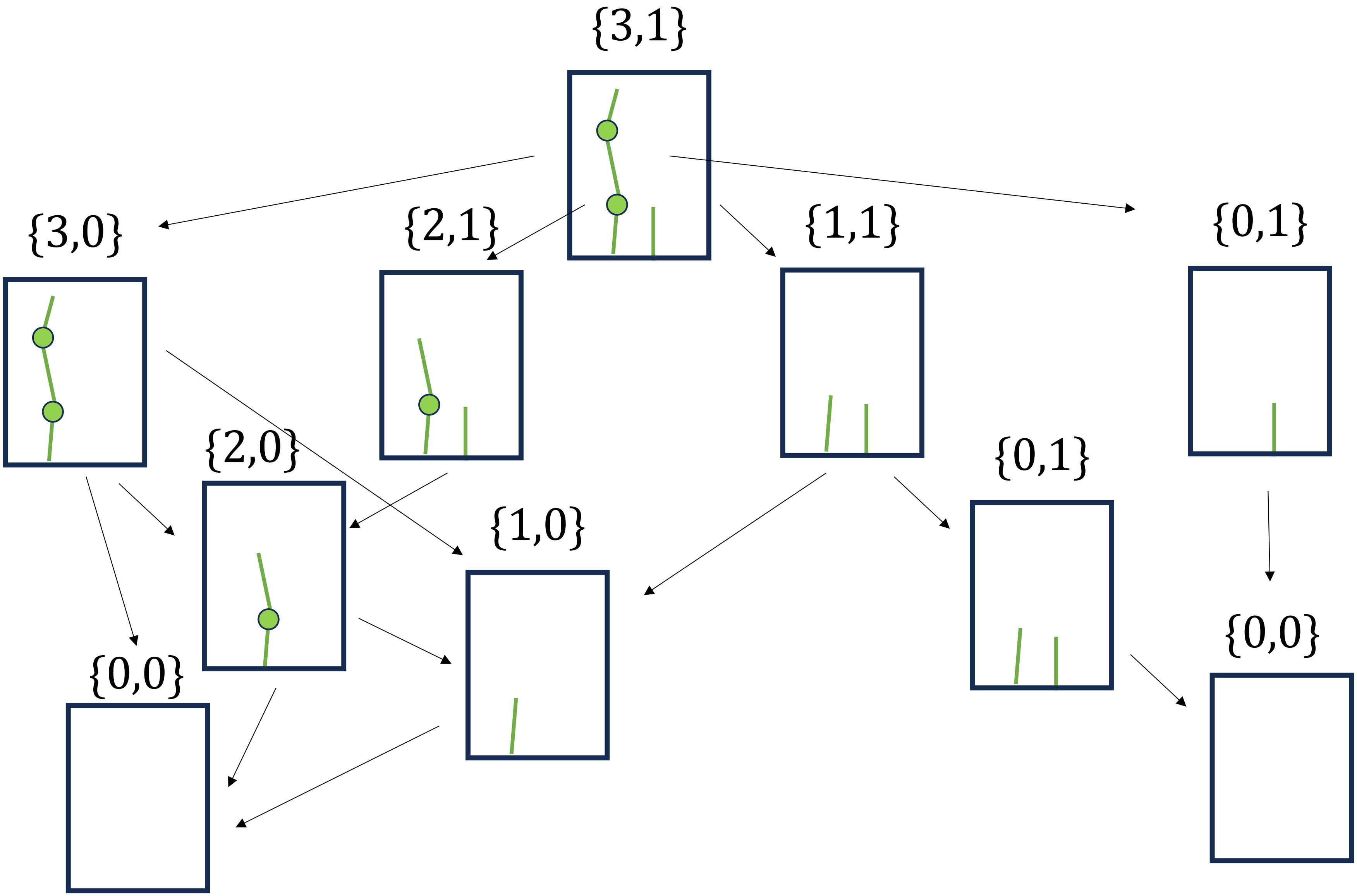

2
1
0
1
1
1
0
0
0
0

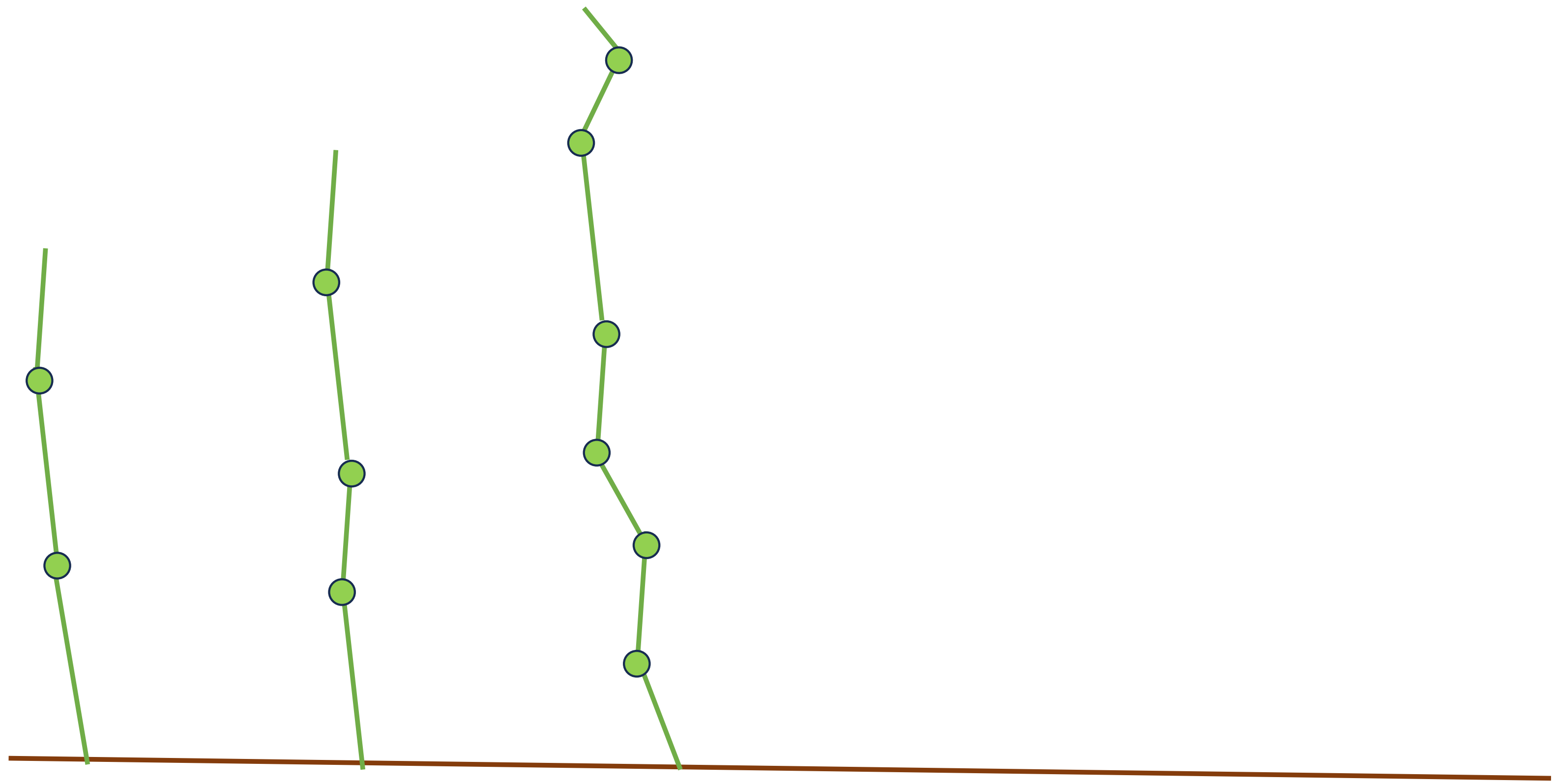
Time for some Hackenbush

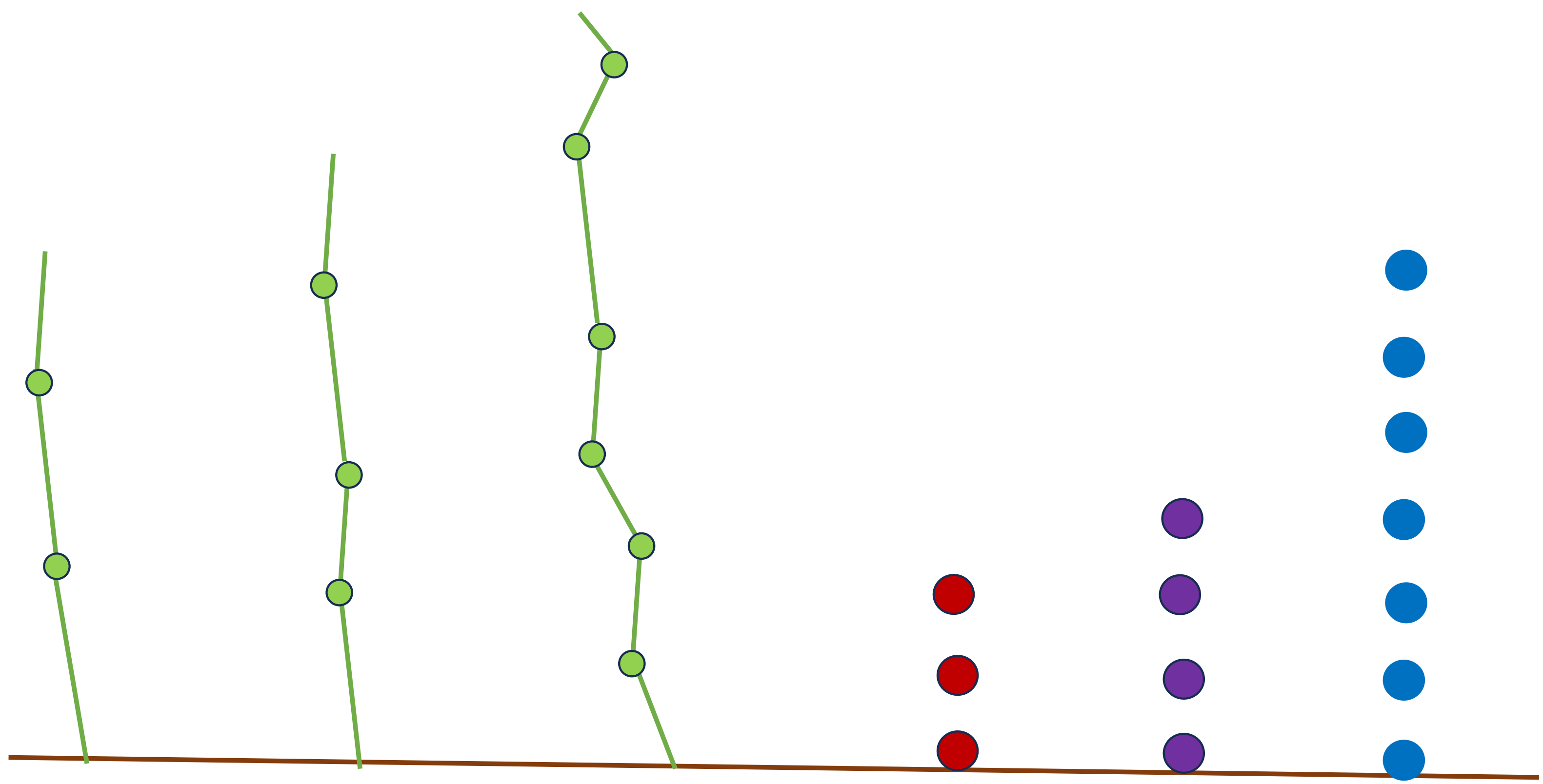


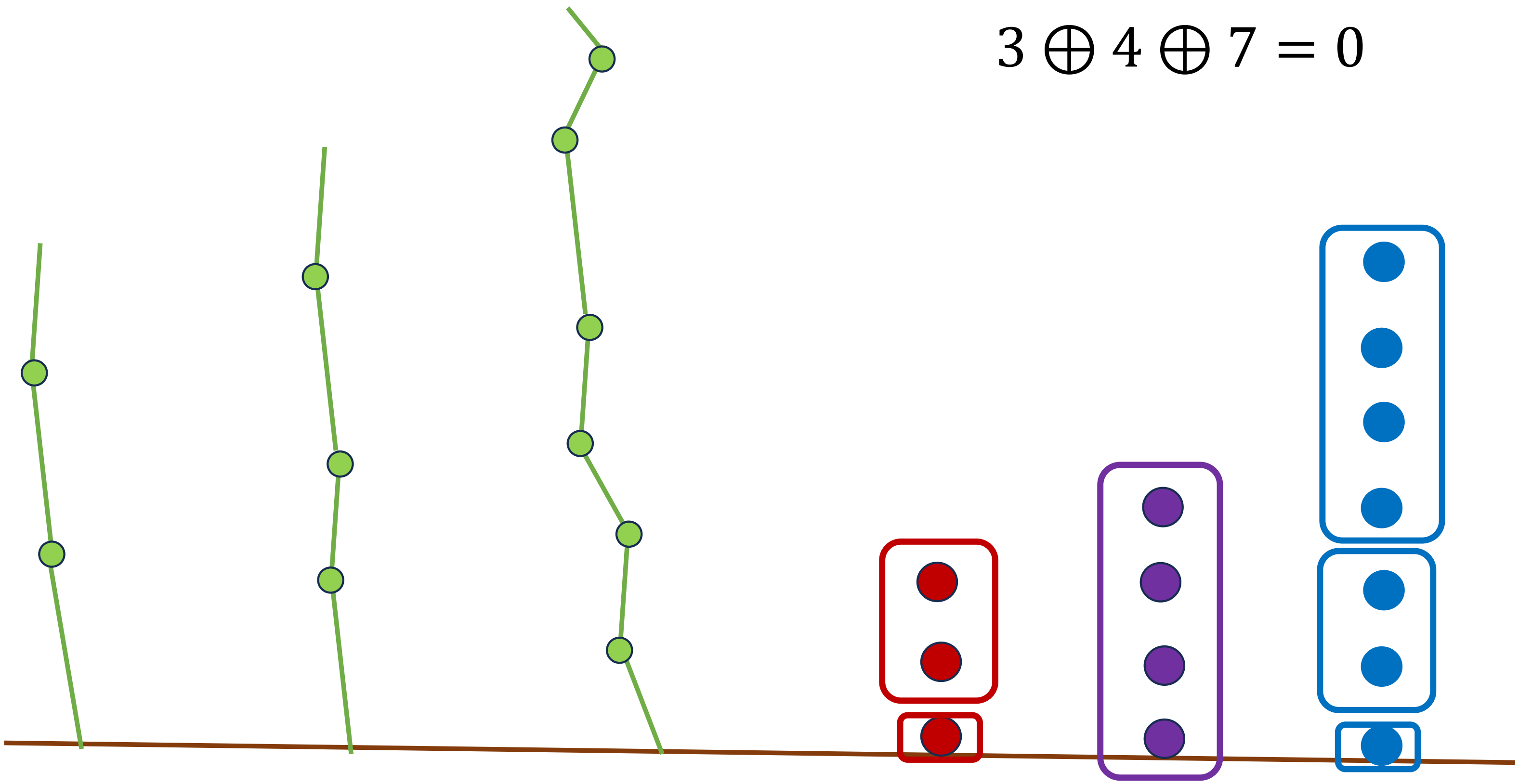


{3,1}
{3,0}
{2,1}
{1,1}
{0,1}


{3,1}
{3,0}
{2,1}
{1,1}
{0,1}
{2,0}
{1,0}
{0,1}
{0,0}
{0,0}






$3 \oplus 4 \oplus 7 = 0$

Wait, is it all Nim?

Wait, is it all Nim?

Sprague – Grundy Theorem

Any finite impartial game is equivalent to a single Nim heap

Wait, is it all Nim?

Sprague – Grundy Theorem

Any finite impartial game is equivalent to a single Nim heap

$$G = * n$$

Who wins Nim with {13, 19, 10}?

$$13 \oplus 19 \oplus 10$$

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$$13 \oplus 19 \oplus 10$$

$$= (8 + 4 + 1) \oplus (16 + 2 + 1) \oplus (8 + 2)$$

Who wins Nim with {13, 19, 10}?

$$13 \oplus 19 \oplus 10$$

$$= (\cancel{8} + 4 + \cancel{1}) \oplus (16 + \cancel{2} + \cancel{1}) \oplus (\cancel{8} + \cancel{2})$$

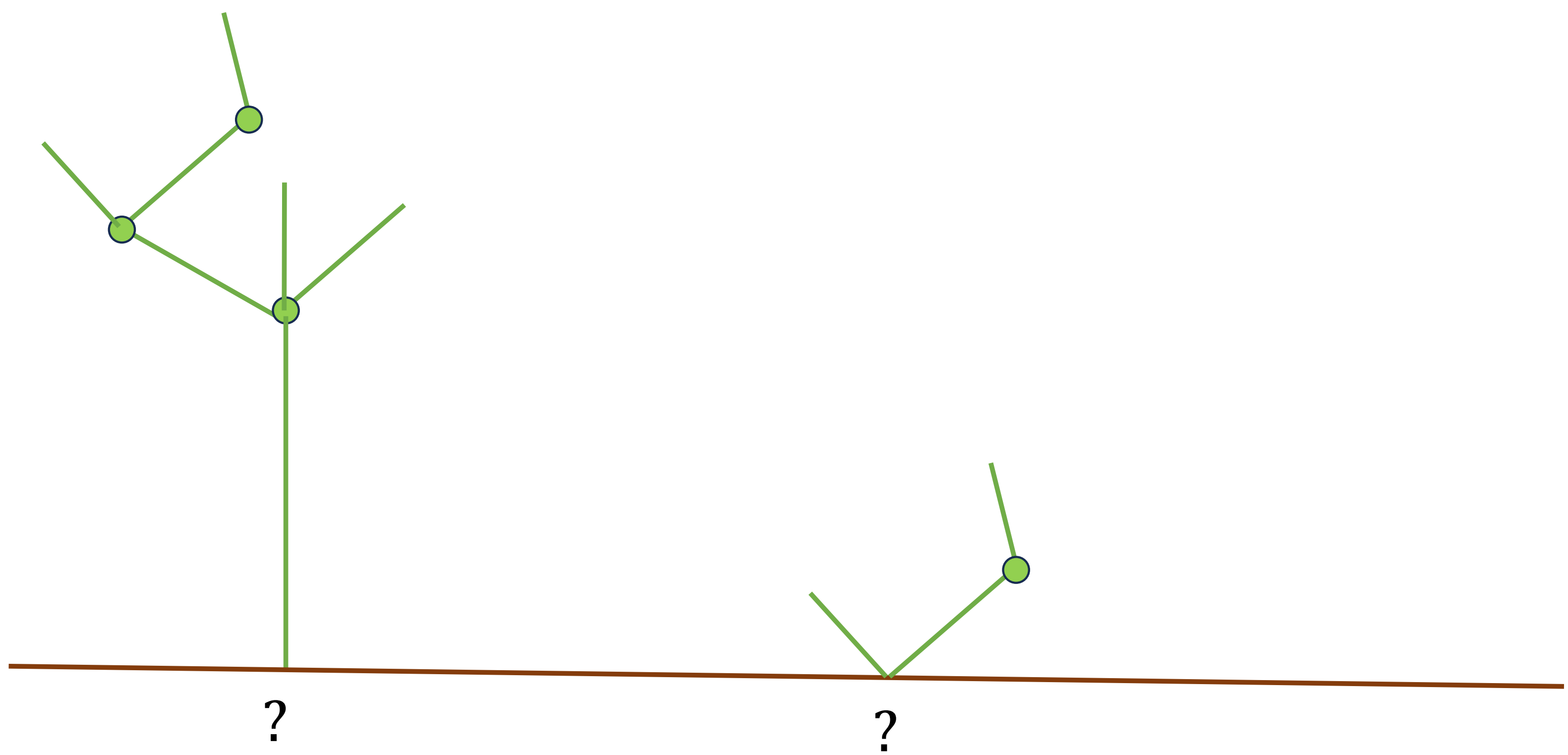
Who wins Nim with {13, 19, 10}?

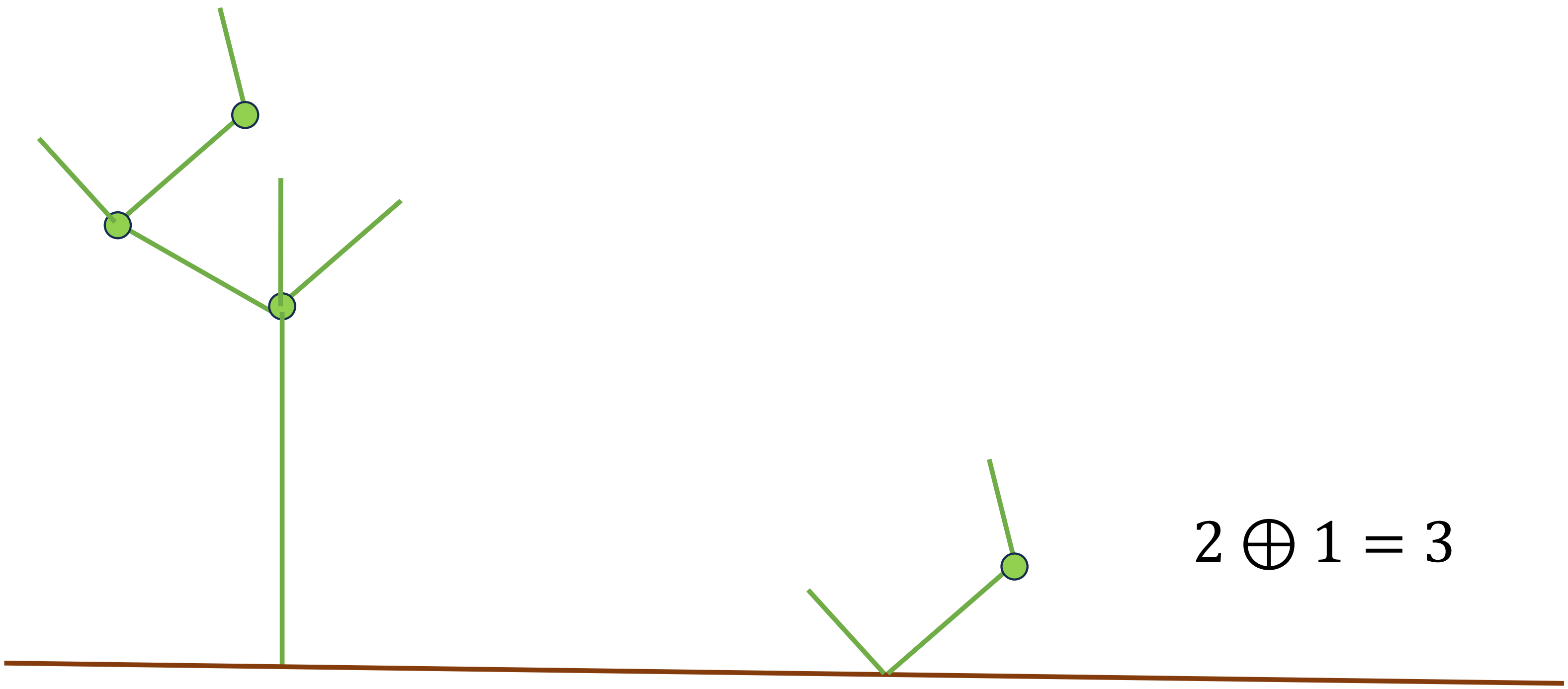
$$13 \oplus 19 \oplus 10$$

$$= (\cancel{8} + 4 + \cancel{1}) \oplus (16 + \cancel{2} + \cancel{1}) \oplus (\cancel{8} + \cancel{2})$$

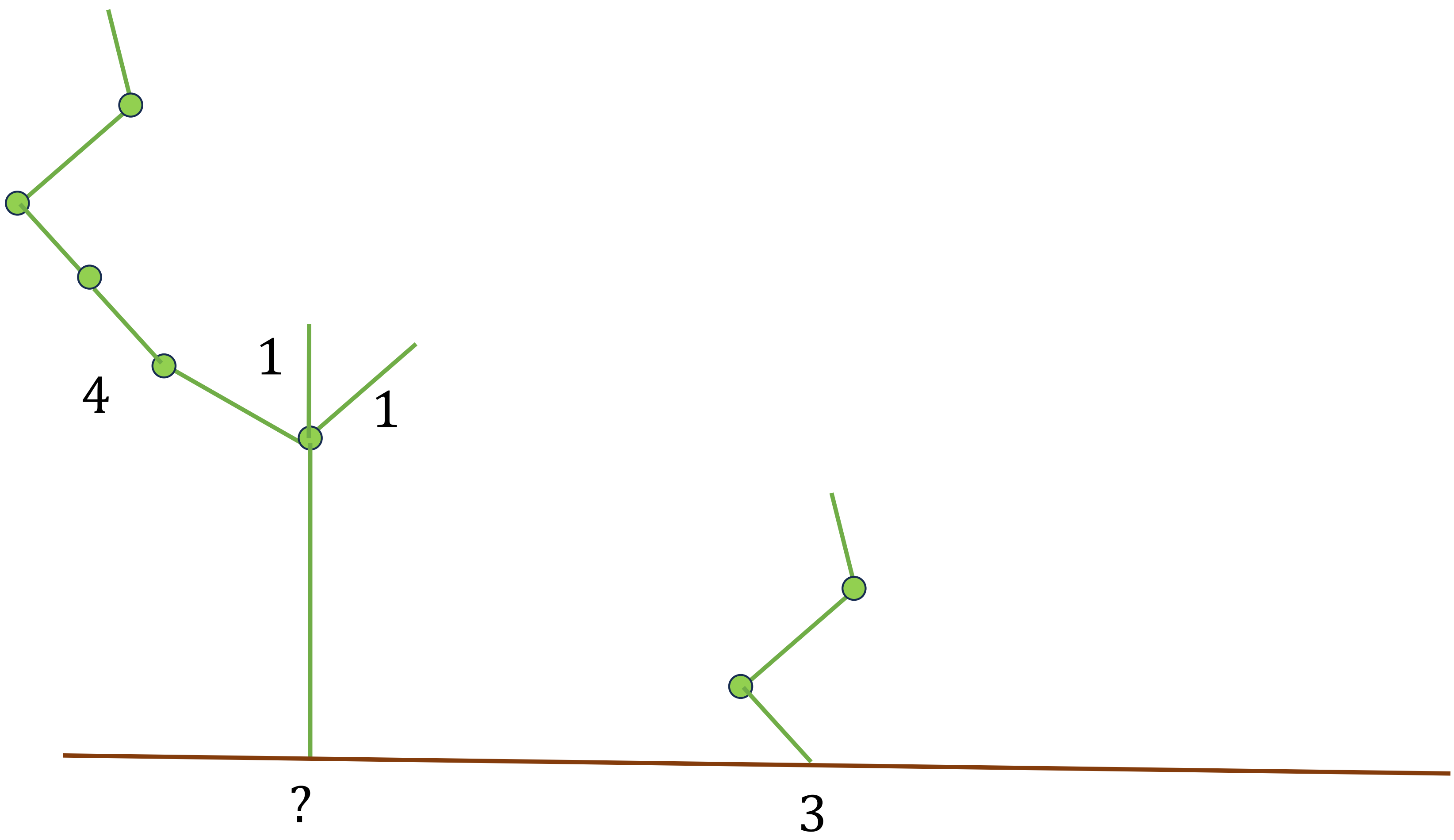
$$4 + 16 = 20$$

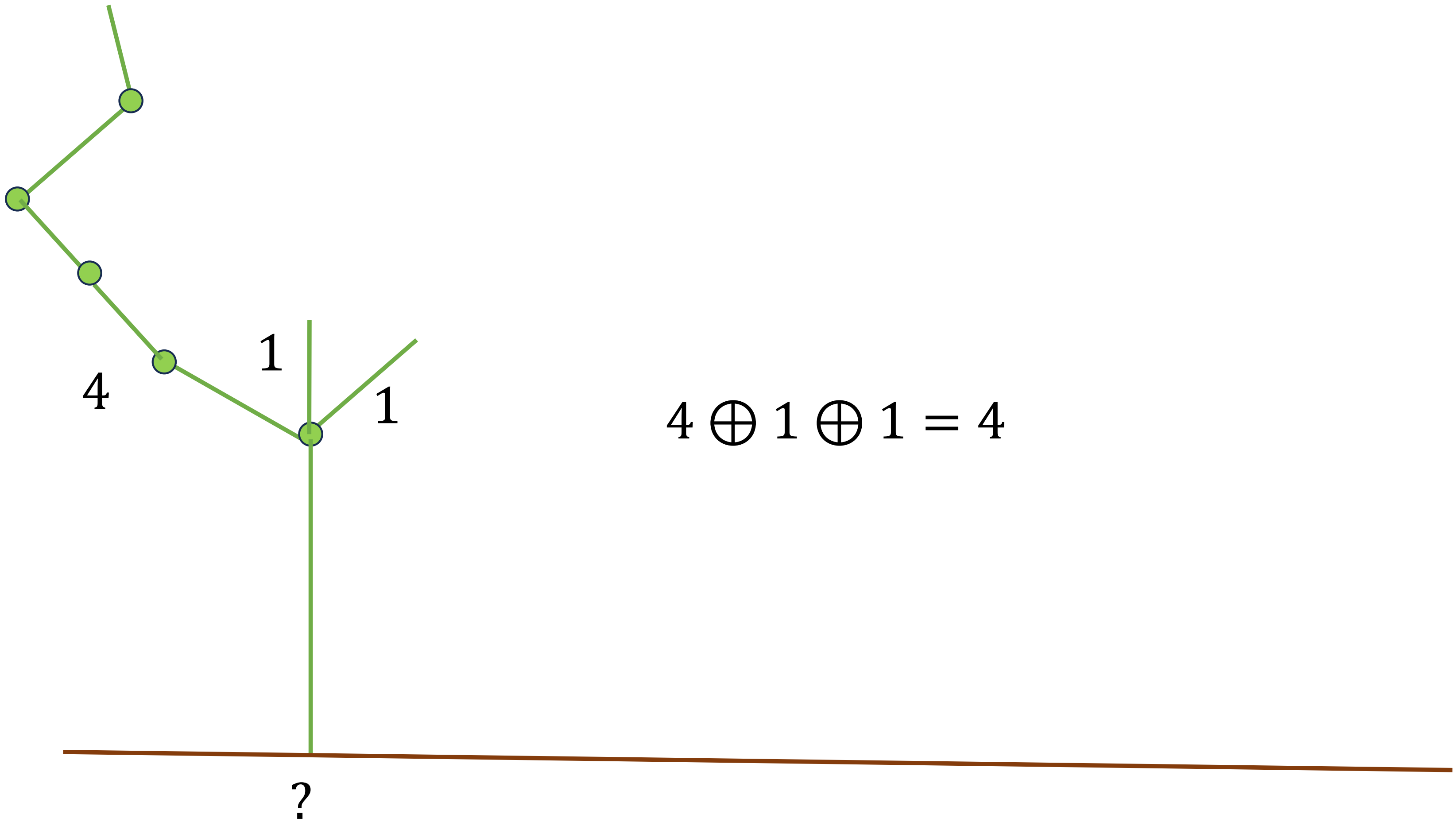
$$G =* 20$$



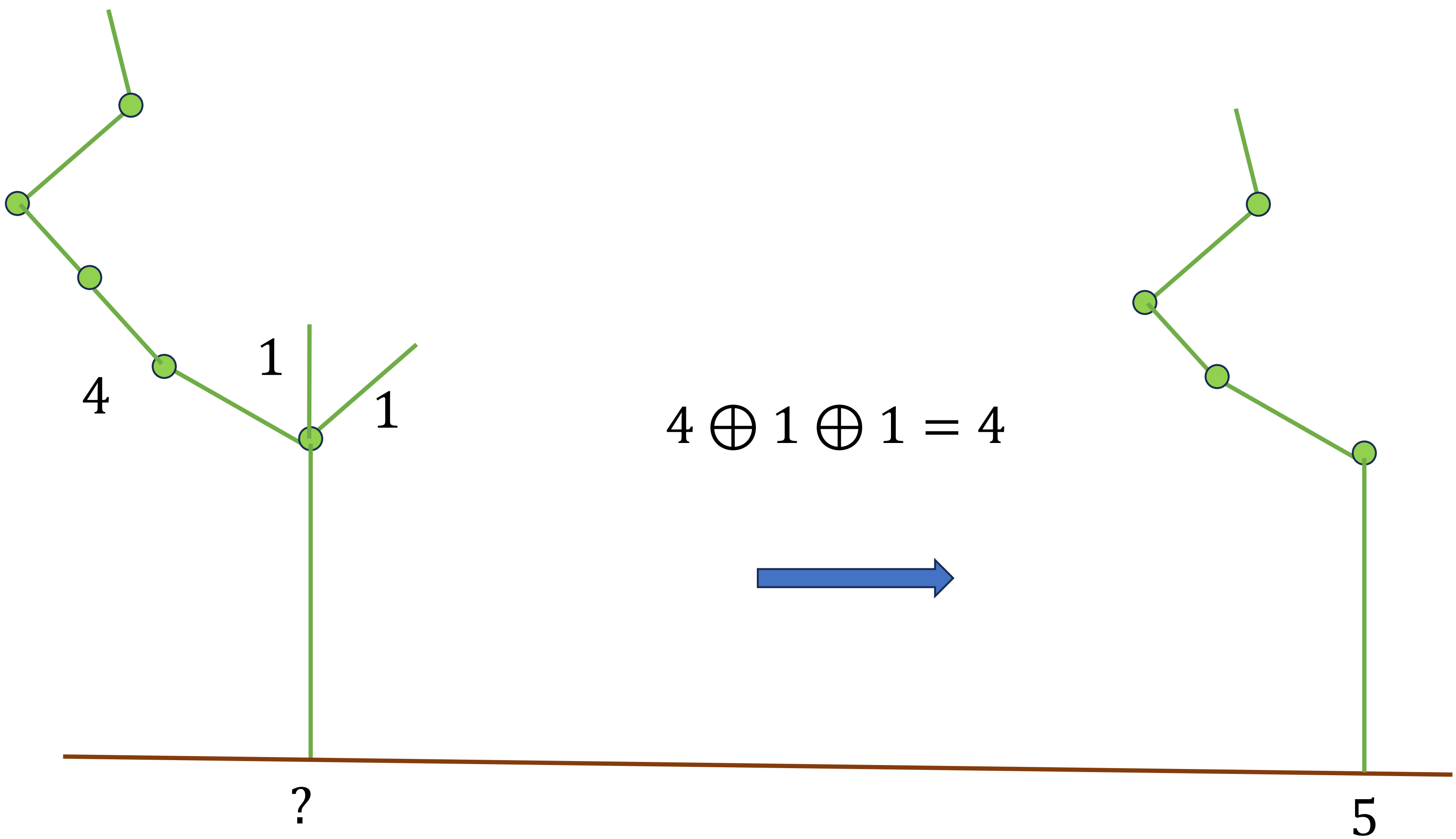


$$2 \oplus 1 = 3$$

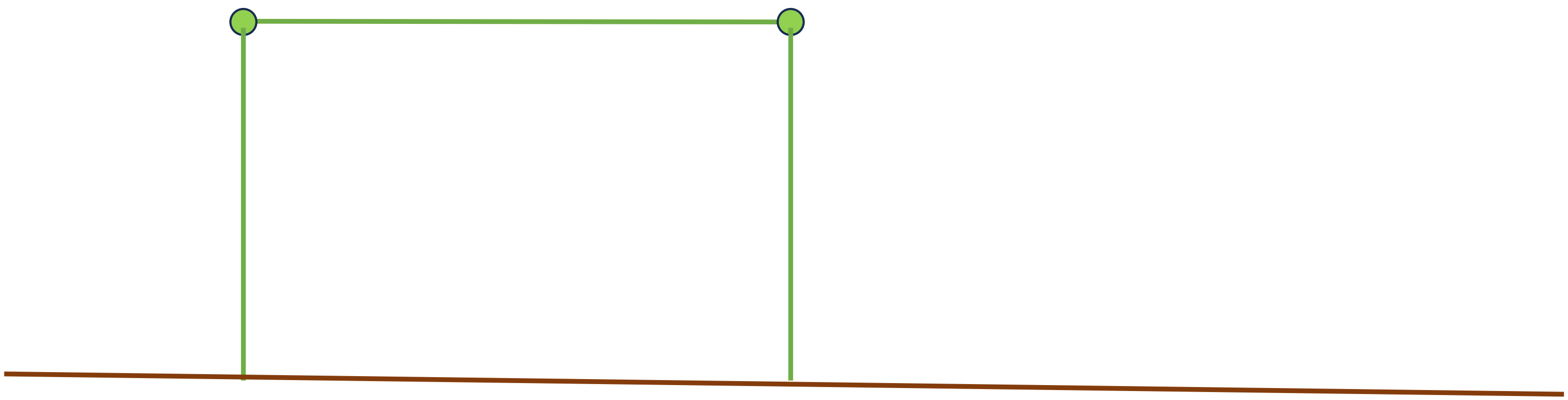

1
4
1
?
3

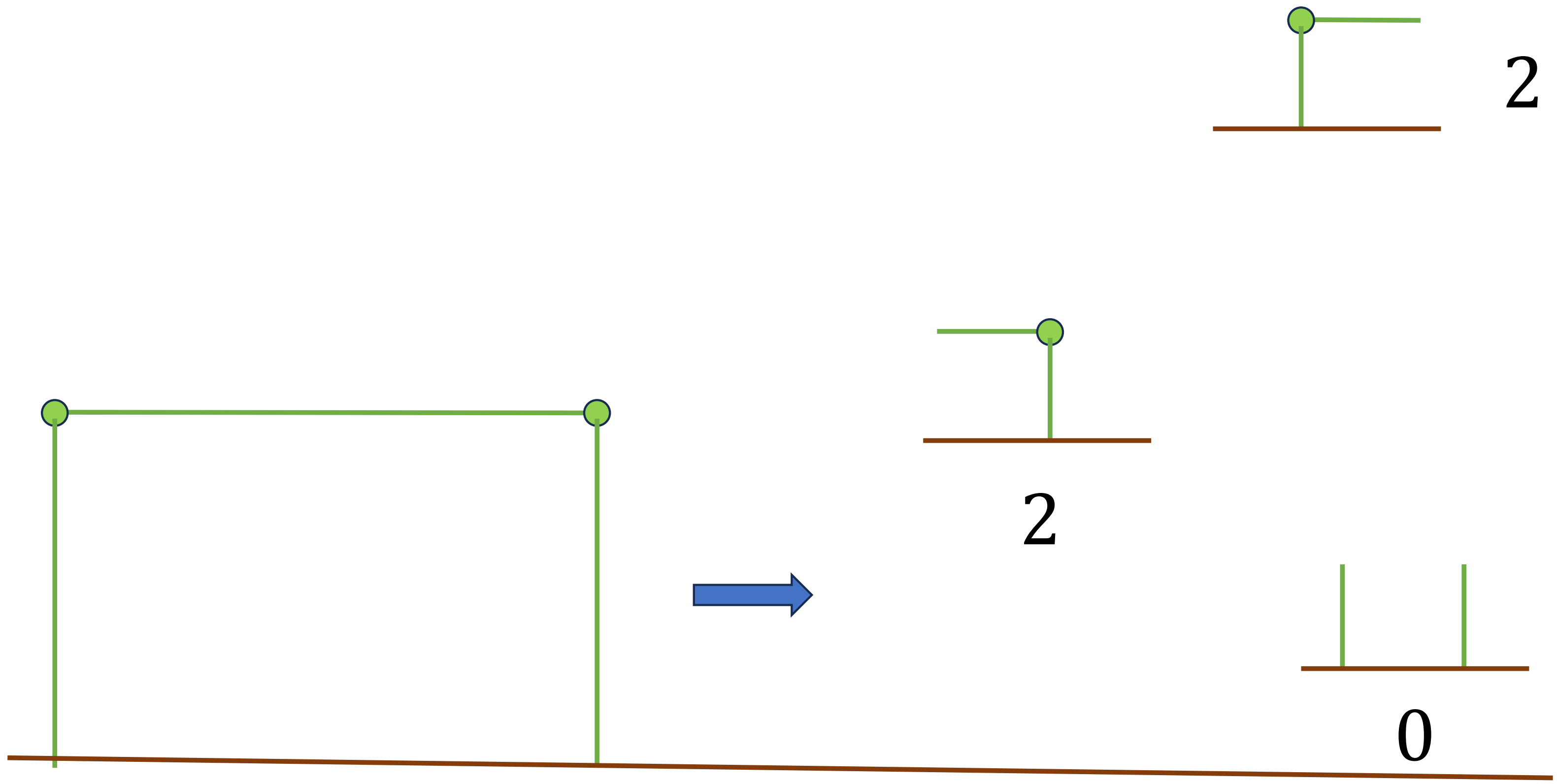


$$4 \oplus 1 \oplus 1 = 4$$

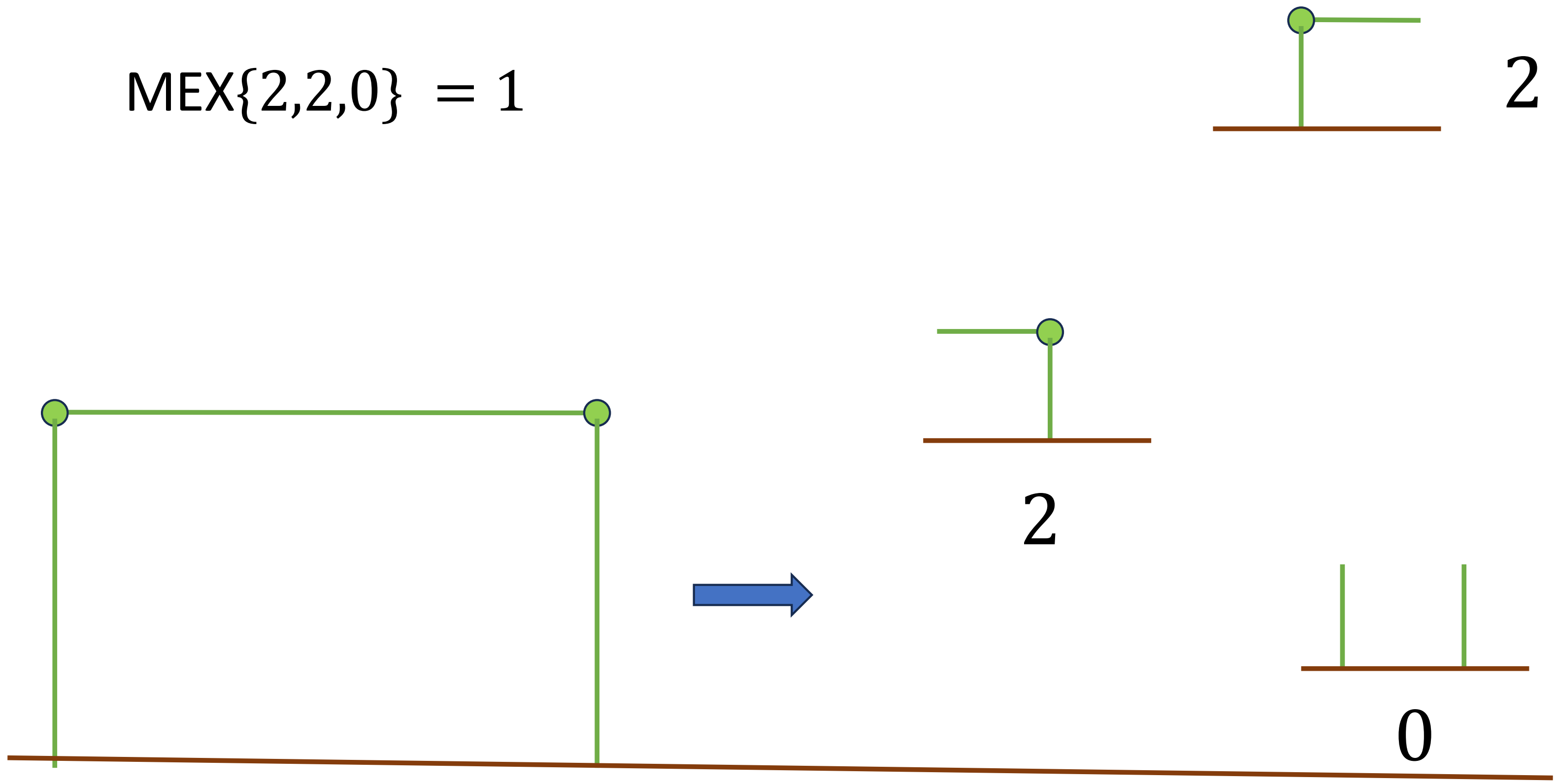


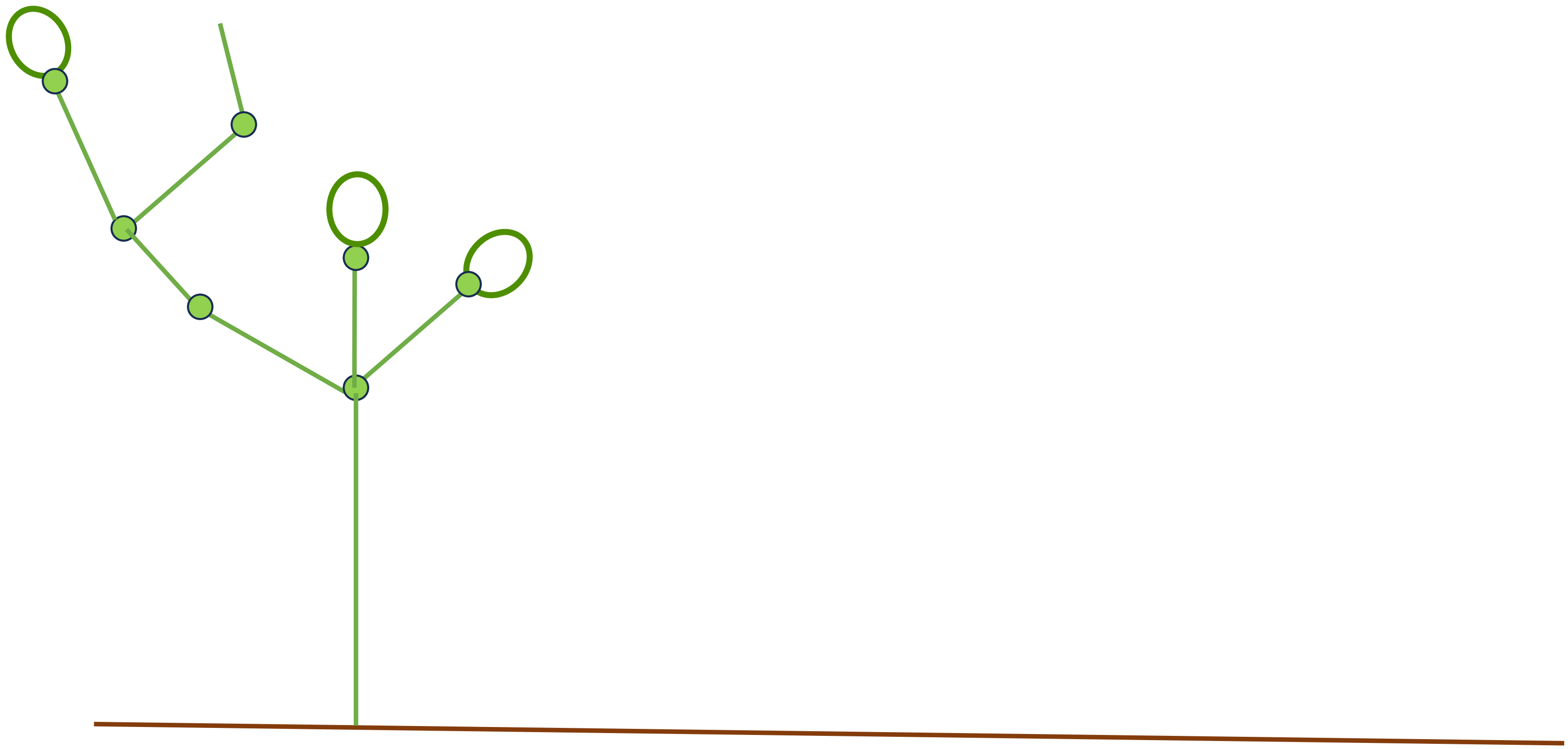
4
1
1
?
$4 \oplus 1 \oplus 1 = 4$
5

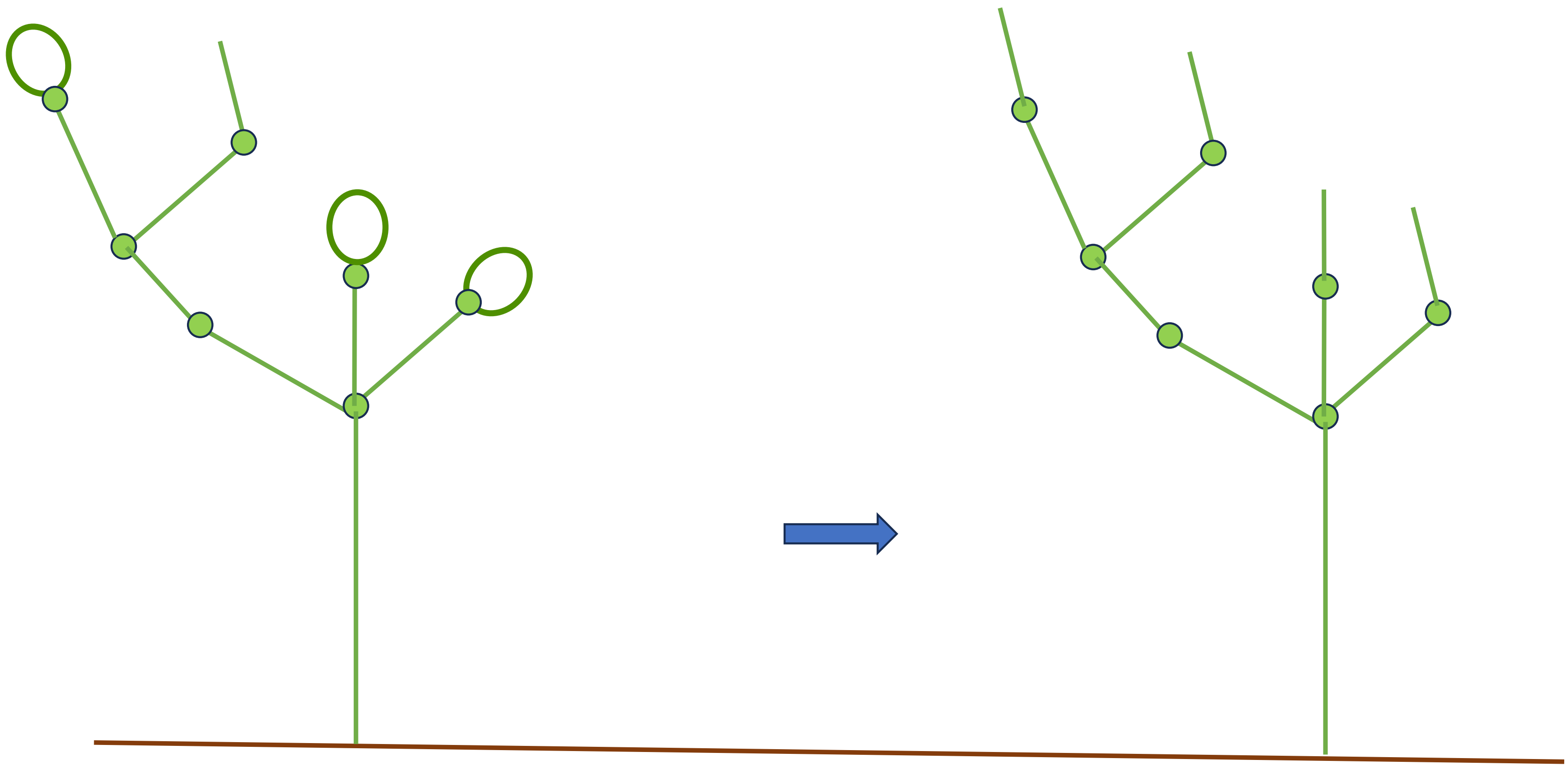


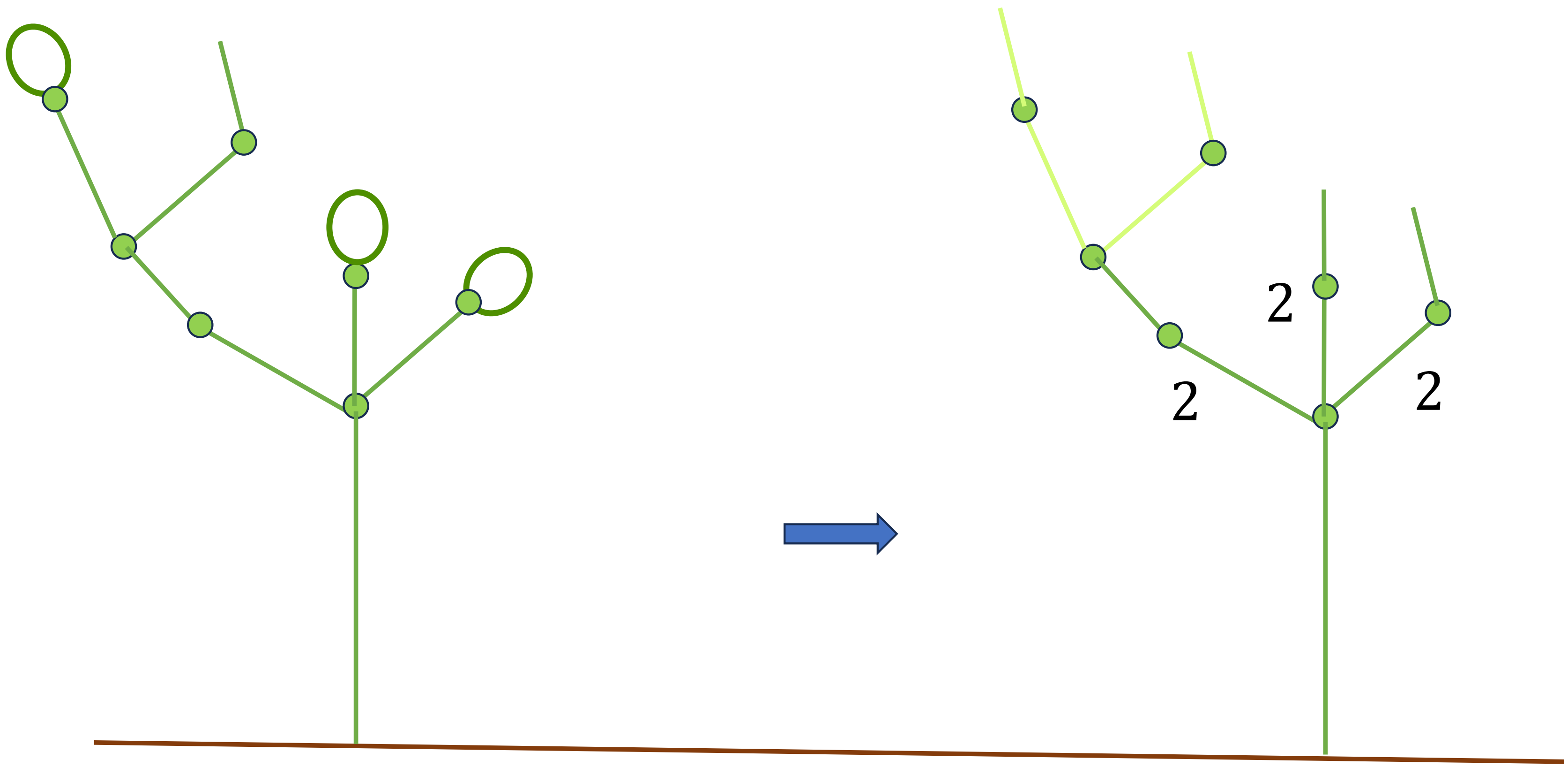

2
2
0

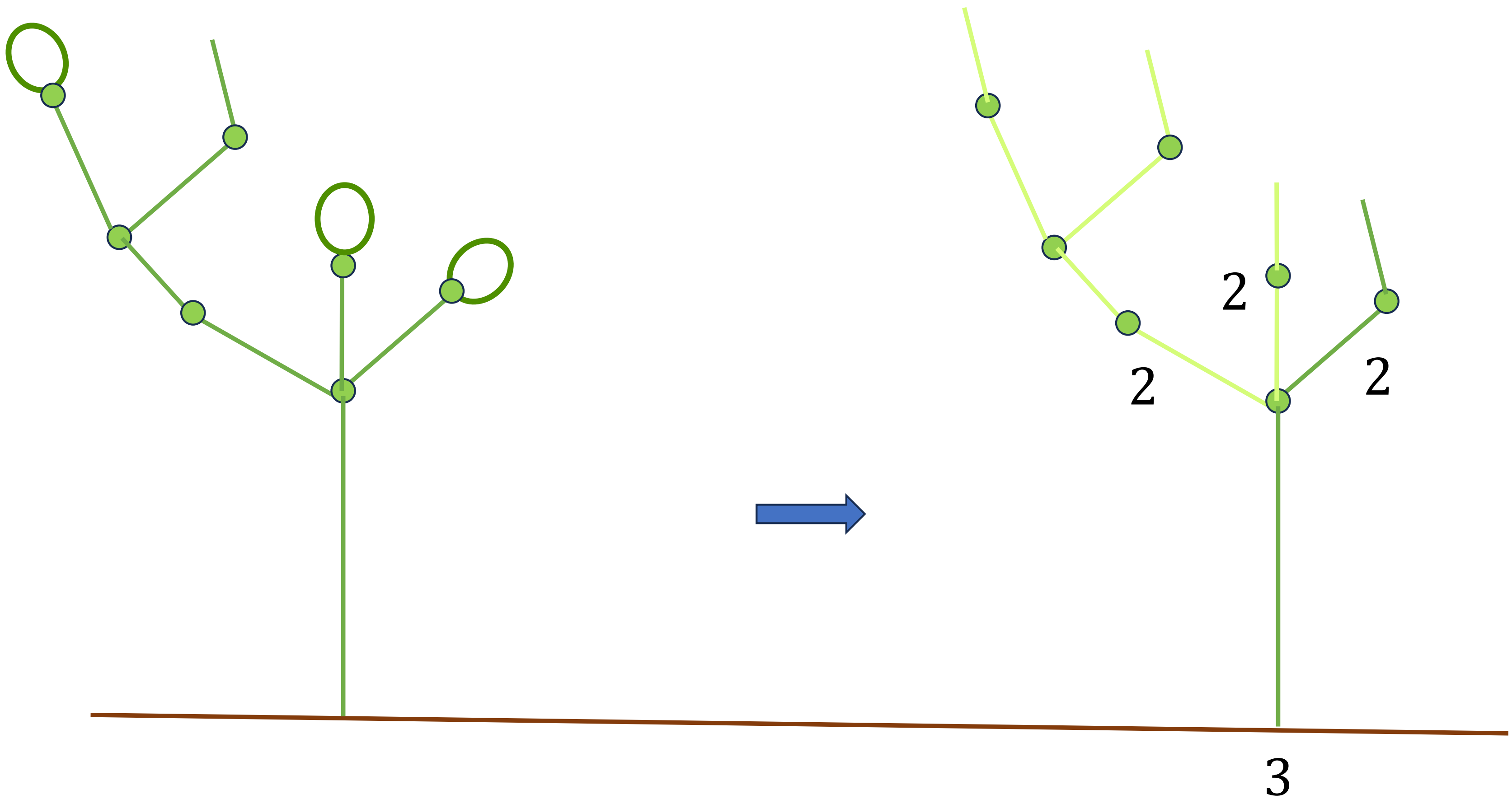
$MEX\{2,2,0\} = 1$

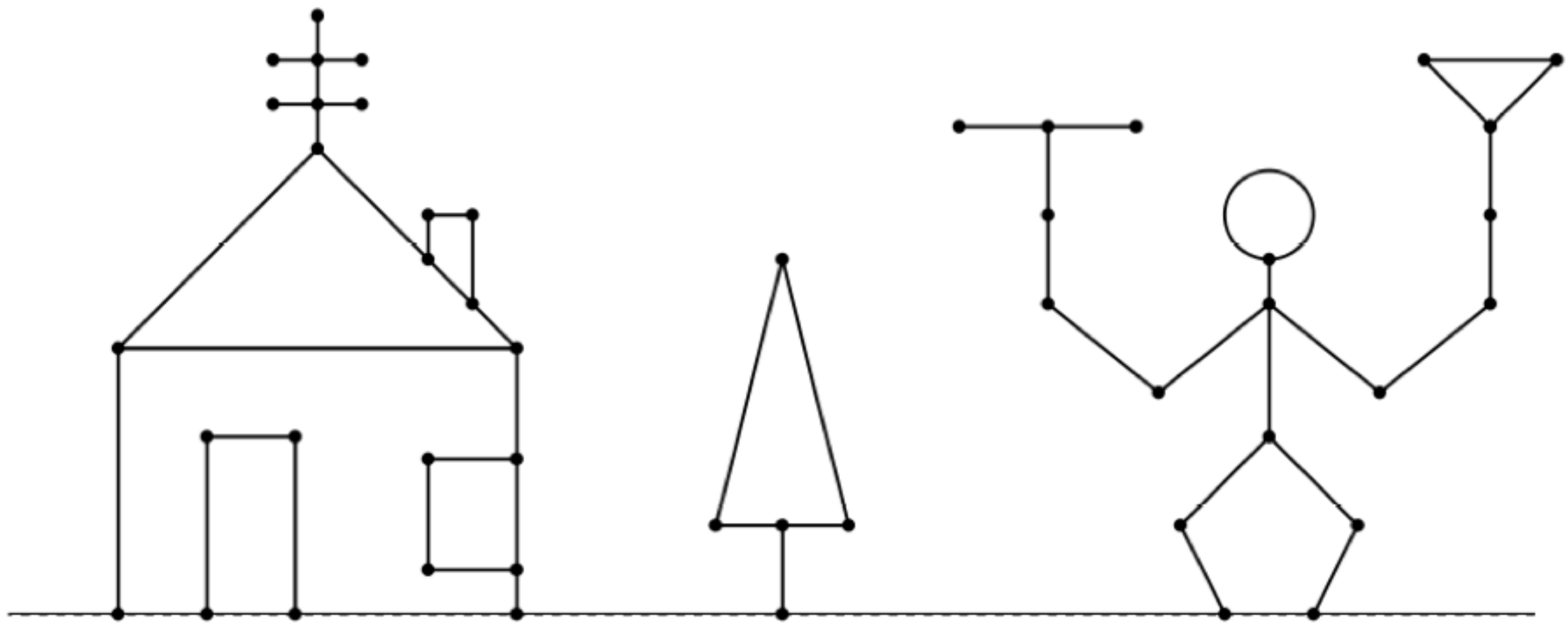


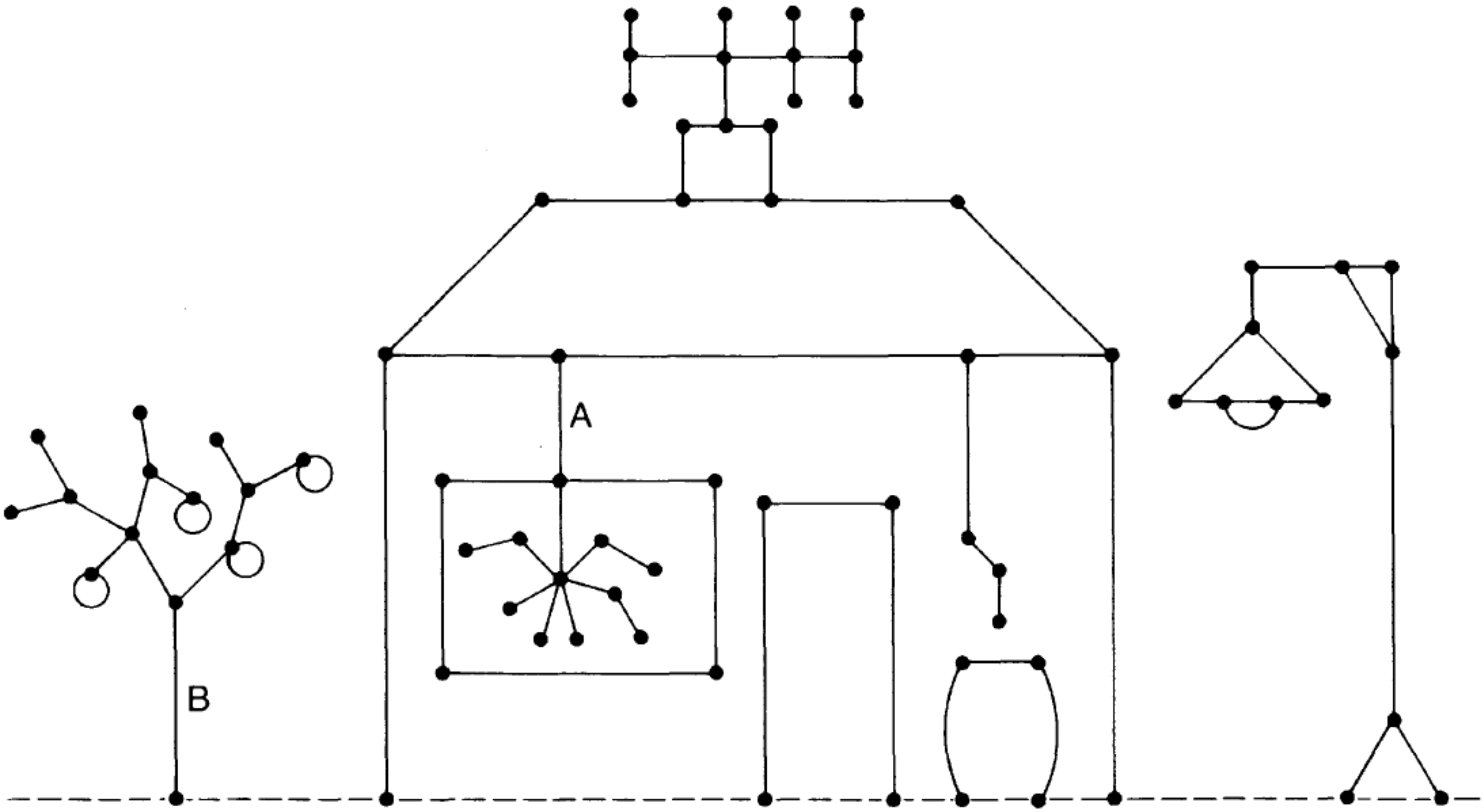





2
2
2


2
2
2
3

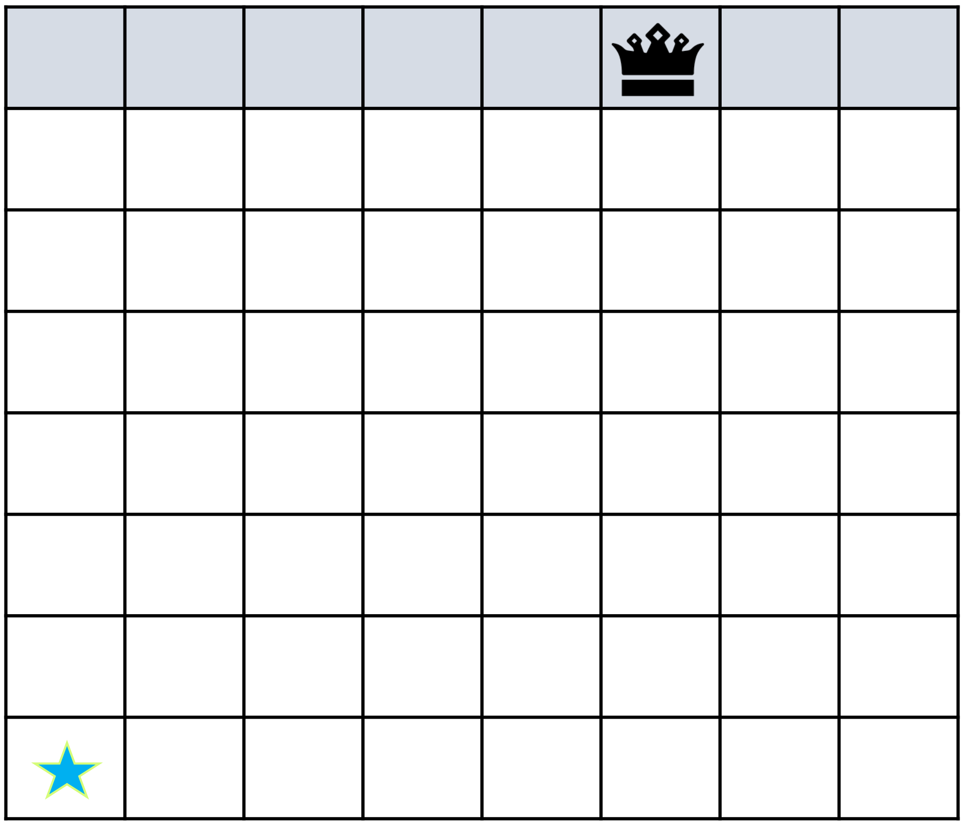




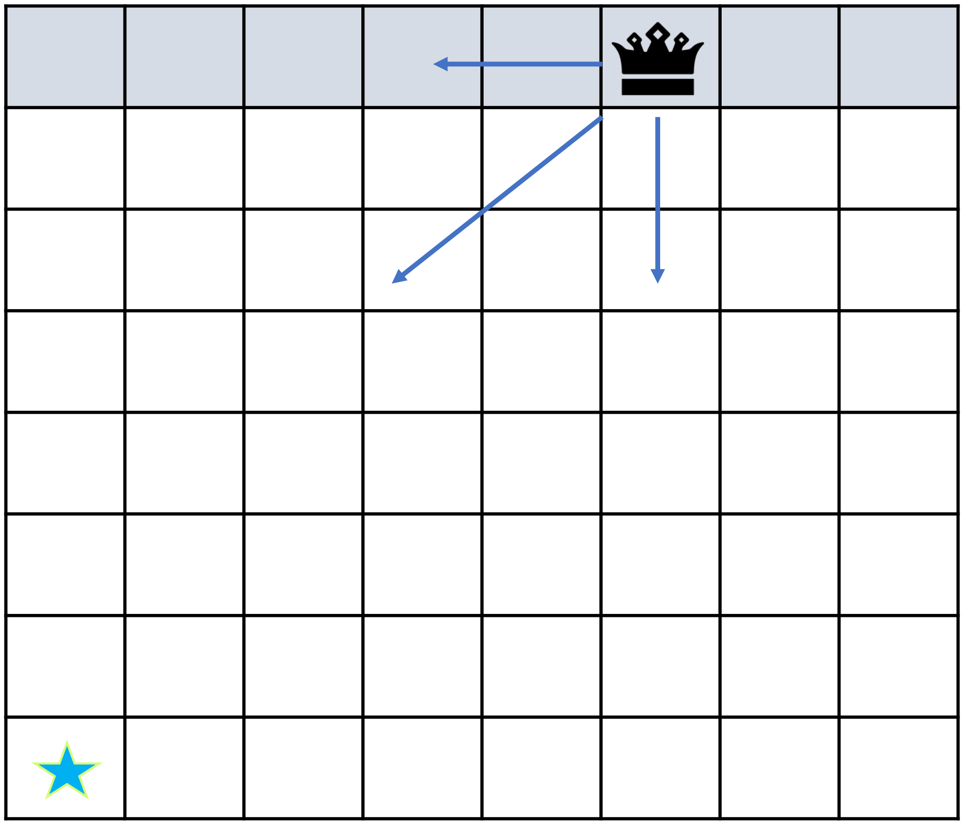
The Hackenbush Homestead

Time for another game!

Corner the Queen



Corner the Queen

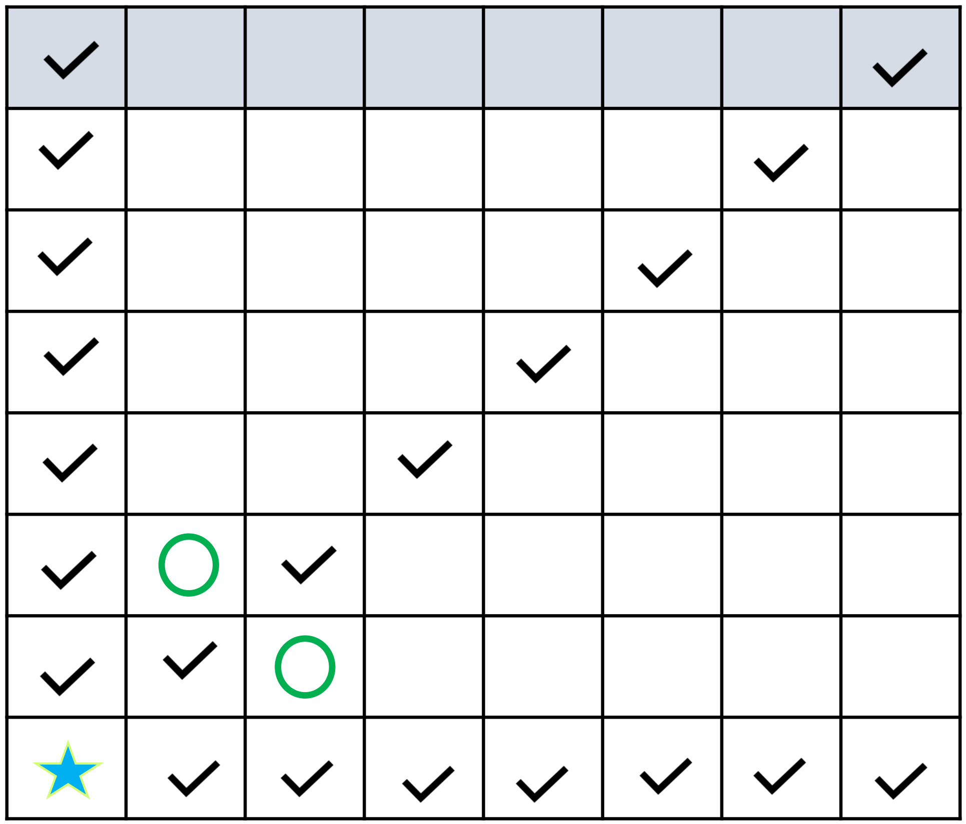


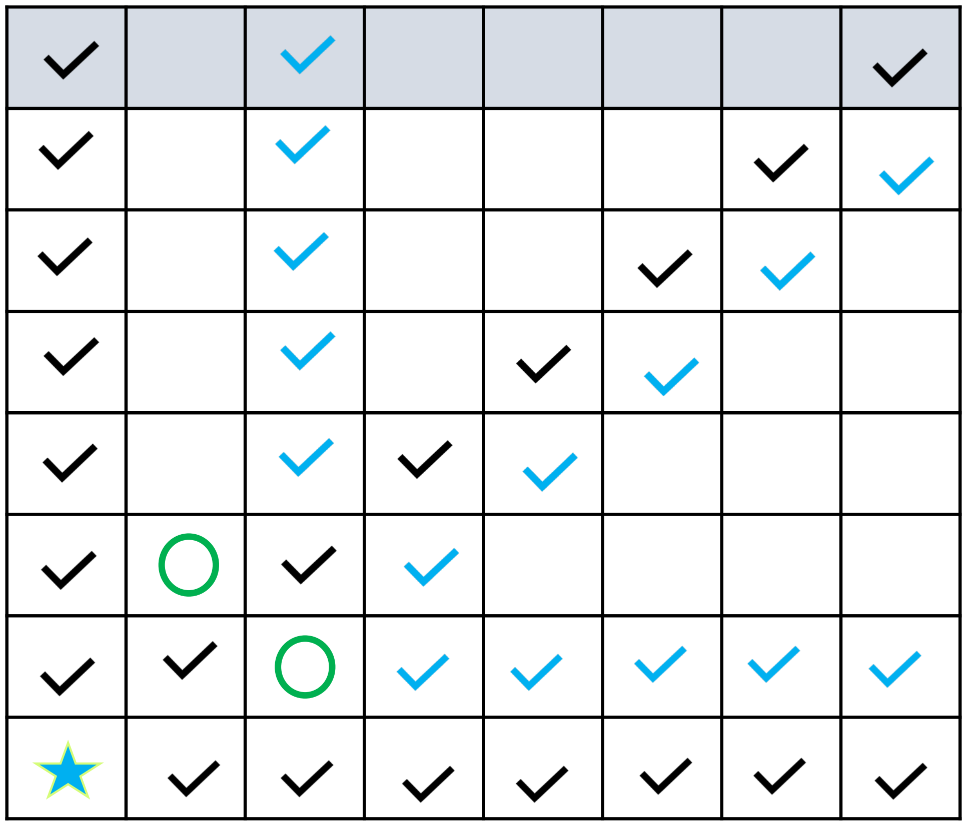
★	✓	✓	✓	✓	✓	✓	✓

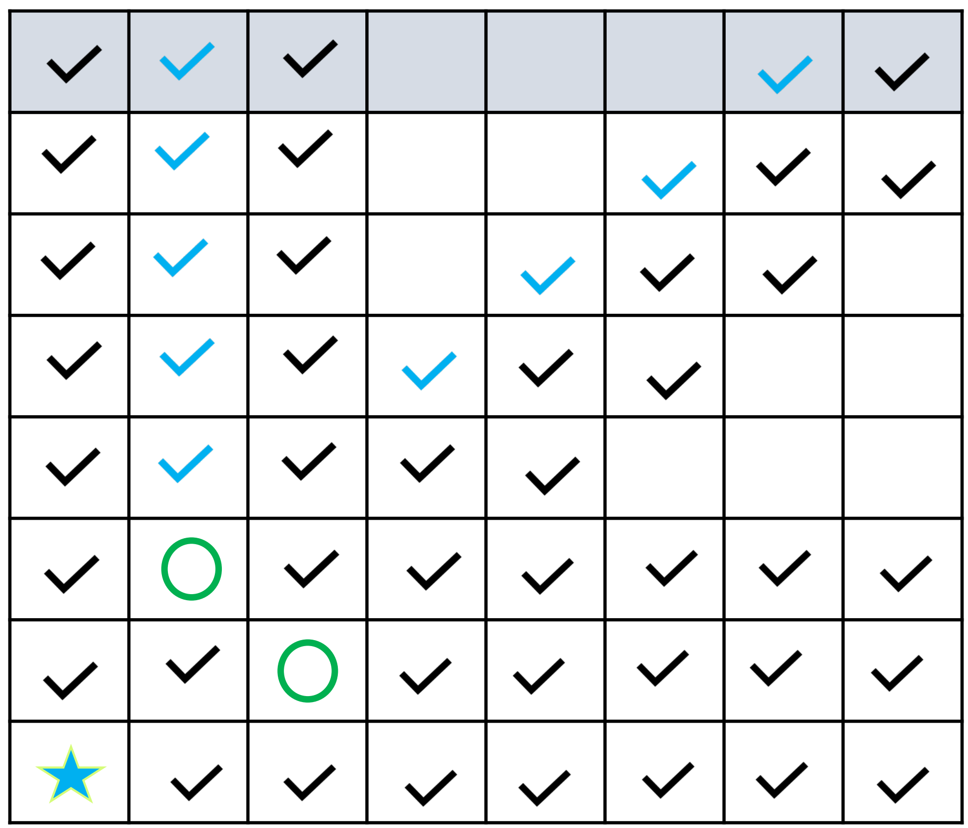
✓							
✓							
✓							
✓							
✓							
✓							
✓							
★	✓	✓	✓	✓	✓	✓	✓

✓							✓
✓						✓	
✓					✓		
✓				✓			
✓			✓				
✓		✓					
✓	✓						
★	✓	✓	✓	✓	✓	✓	✓

✓							✓
✓						✓	
✓					✓		
✓				✓			
✓			✓				
✓	?	✓					
✓	✓	?					
★	✓	✓	✓	✓	✓	✓	✓

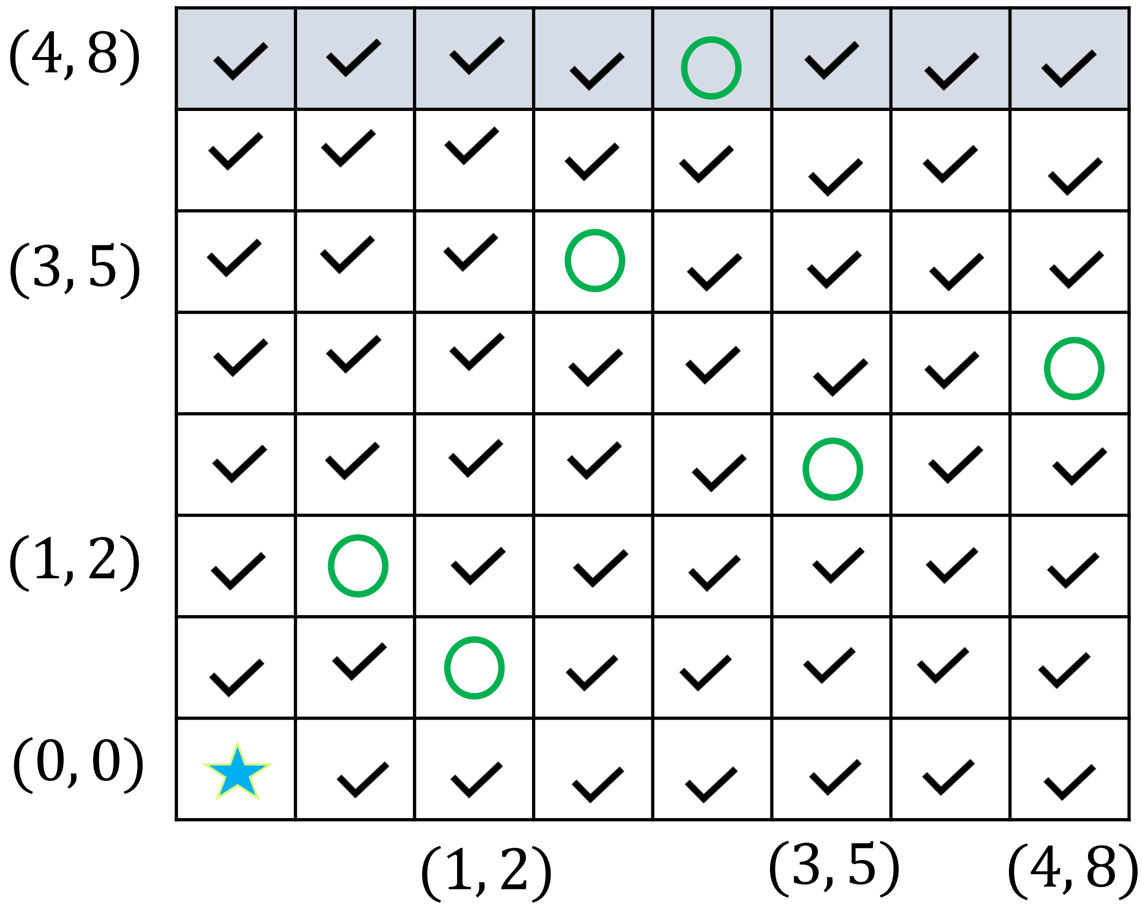


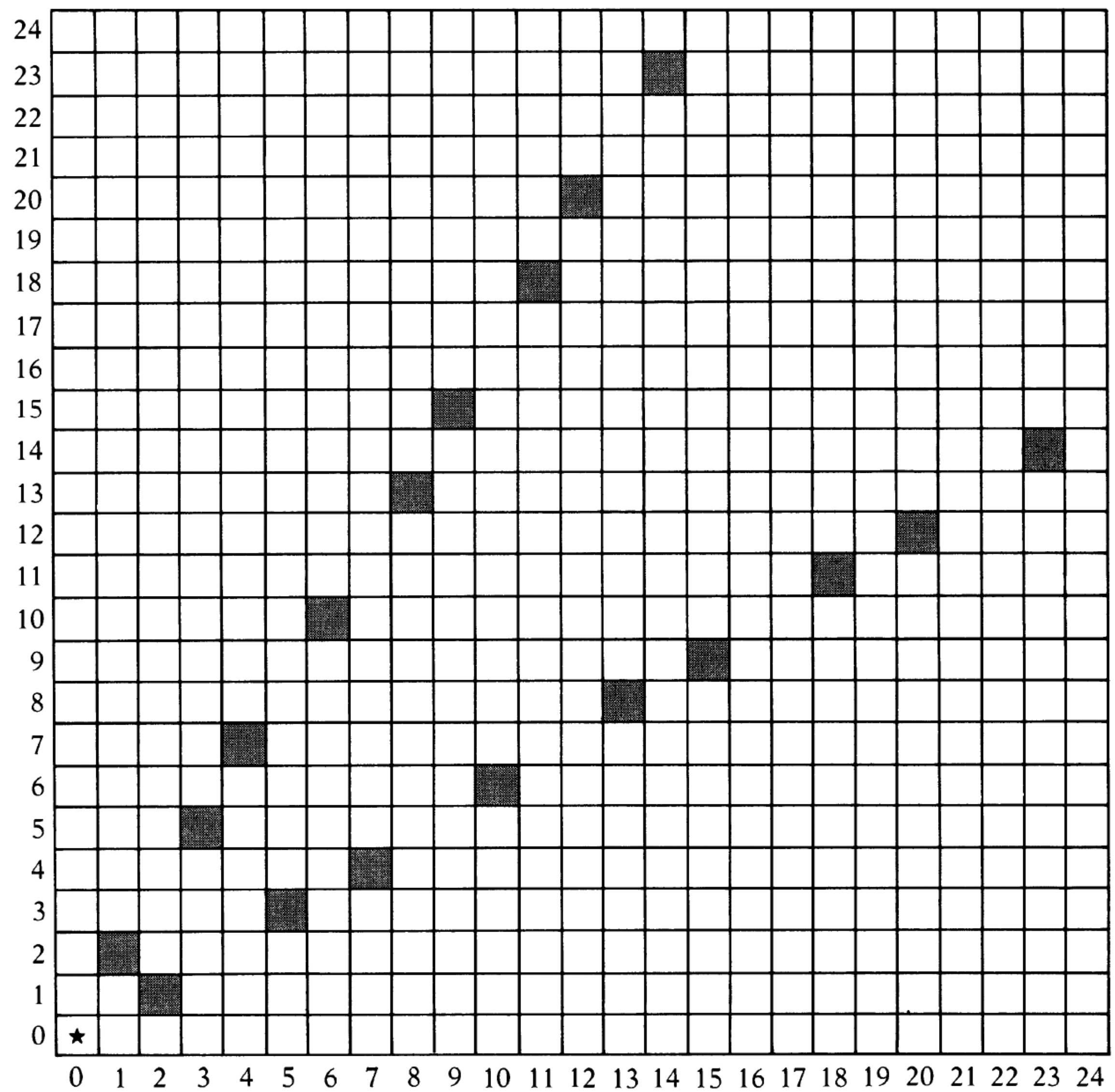




✓	✓	✓				✓	✓
✓	✓	✓			✓	✓	✓
✓	✓	✓	?	✓	✓	✓	
✓	✓	✓	✓	✓	✓		
✓	✓	✓	✓	✓	?		
✓	○	✓	✓	✓	✓	✓	✓
✓	✓	○	✓	✓	✓	✓	✓
★	✓	✓	✓	✓	✓	✓	✓

✓	✓	✓	✓	○	✓	✓	✓
✓	✓	✓	✓	✓	✓	✓	✓
✓	✓	✓	○	✓	✓	✓	✓
✓	✓	✓	✓	✓	✓	✓	○
✓	✓	✓	✓	✓	○	✓	✓
✓	○	✓	✓	✓	✓	✓	✓
✓	✓	○	✓	✓	✓	✓	✓
★	✓	✓	✓	✓	✓	✓	✓

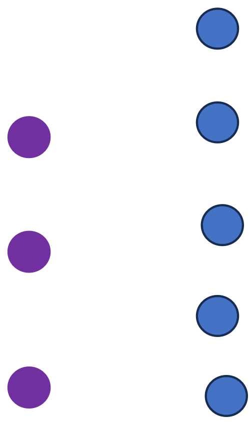

(4, 8)
(3, 5)
(1, 2)
(0, 0)
(1, 2)
(3, 5)
(4, 8)



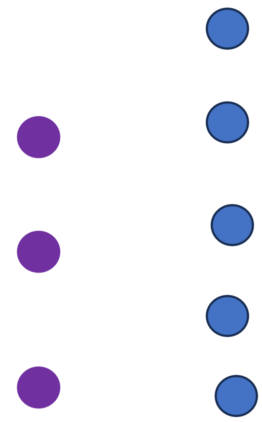
24
23
22
21
20
19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

Wythoff Nim

Played with 2 rows of counters



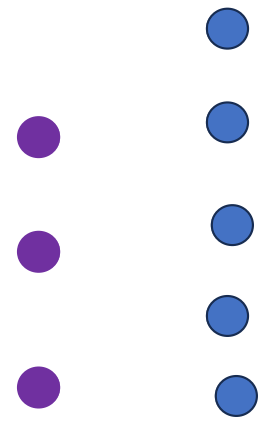
Wythoff Nim



Played with 2 rows of counters

Can take from both rows if take same number from both

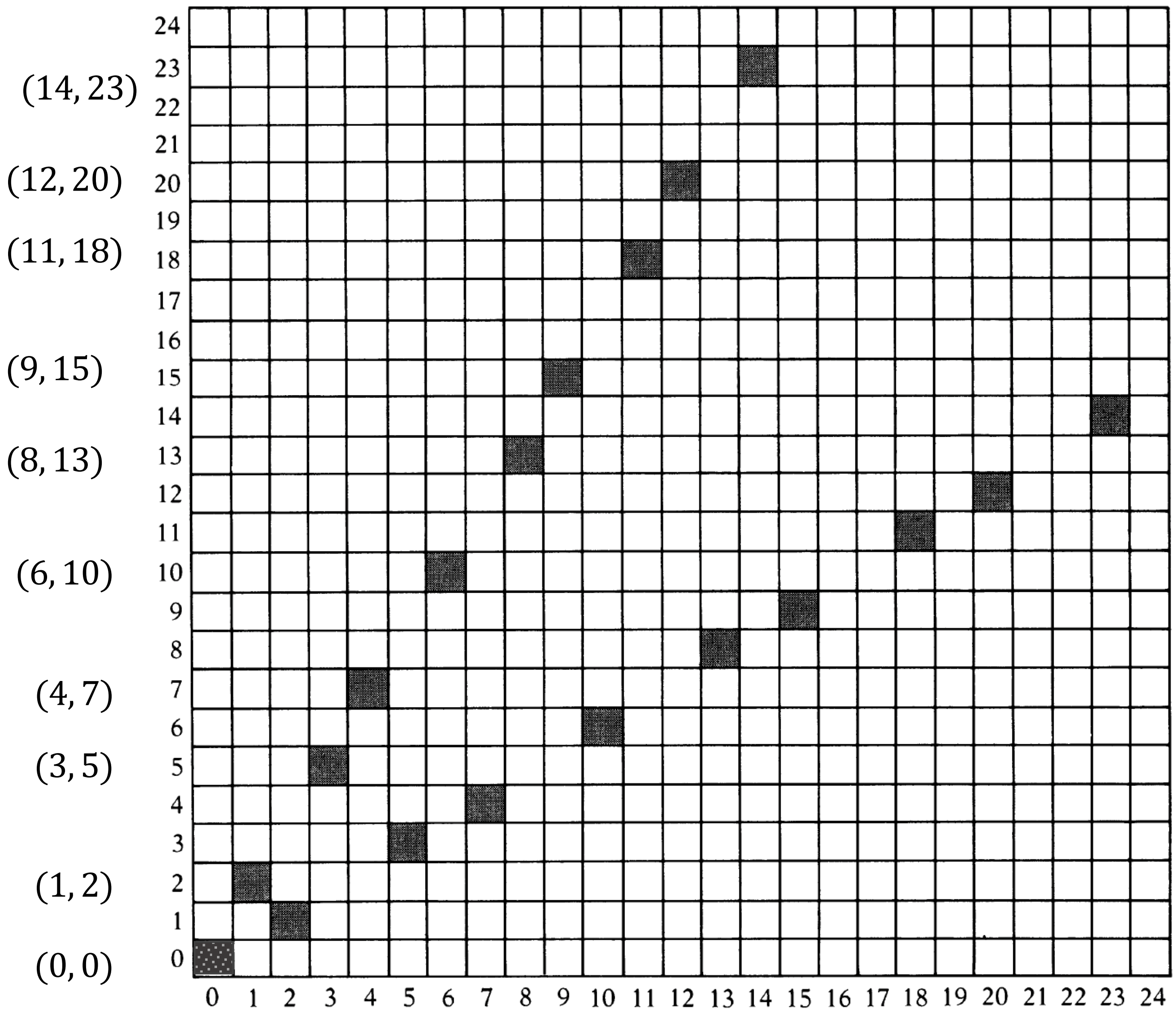
Wythoff Nim



Played with 2 rows of counters

Can take from both rows if take same number from both

Take at least one counter – can empty a row


(14, 23)
(12, 20)
(11, 18)
(9, 15)
(8, 13)
(6, 10)
(4, 7)
(3, 5)
(1, 2)
(0, 0)

1	3	4	6	8	9	11	12	14
2	**5**	**7**	**10**	**13**	**15**	**18**	**20**	**23**

Fibonnacci numbers appear

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

1	3	4	6	8	9	11	12	14
2	**5**	**7**	**10**	**13**	**15**	**18**	**20**	**23**

$$(1, 2), (3, 5), (8, 13), \ldots$$

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$(1, 2), (3, 5), (8, 13), \ldots$

$(4, 7), (11, 18), \ldots$

$(6, 10), (16, 26), \ldots$

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$(4, 7), (11, 18), \ldots$

$(6, 10), (16, 26), \ldots$

Fibonnacci numbers appear

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

A	1	3	4	6	8	9	11	12	14
B	**2**	**5**	**7**	**10**	**13**	**15**	**18**	**20**	**23**

$(1, 2), (3, 5), (8, 13), \ldots$

$(4, 7), (11, 18), \ldots$

$(6, 10), (16, 26), \ldots$

Determining a Safe Play

Any natural number can be written uniquely as a sum of non-consecutive Fibonacci (Pingala) numbers

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For example, $17 = 13 + 3 + 1$

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21	13	8	5	3	2	1

$$1, 2, 3, 5, 8, 13, 21, \ldots$$

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For example, $17 = 13 + 3 + 1$

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0	1	0	0	1	0	1

$1, 2, 3, 5, 8, 13, 21, \ldots$

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For example, $17 = 13 + 3 + 1$

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0	1	0	0	1	0	1

$1, 2, 3, 5, 8, 13, 21, \ldots$

Determining a Safe Play

$(1, 2), (3, 5), (8, 13), \ldots$

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Determining a Safe Play

$(1,2), (3,5), (8,13), \ldots$

$(1,10), (100,1000), (1000,10000), \ldots$

21	13	8	5	3	2	1

Determining a Safe Play

$(4, 7), (11, 18), \ldots$

21	13	8	5	3	2	1

Determining a Safe Play

$(4, 7), (11, 18), \ldots$

$(101,1010), (10100,101000), \ldots$

21	13	8	5	3	2	1

Determining a Safe Play

$(6, 10), (16, 26), \ldots$

$(101{,}1010), (10100{,}101000), \ldots$

21	13	8	5	3	2	1

Determining a Safe Play

A	1	3	4	6	8	9	11	12	14
B	2	5	7	10	13	15	18	20	23

$(1, 2)$, $(3, 5)$, $(4,7)$, $(6,10)$, $(8, 13)$, ...

$(1, 10), (100, 1000), (101,1010), (1001,10010), (10000, 100000), \ldots$

21	13	8	5	3	2	1

Determining a Safe Play

A	1	3	4	6	8	9	11	12	14
B	2	5	7	10	13	15	18	20	23

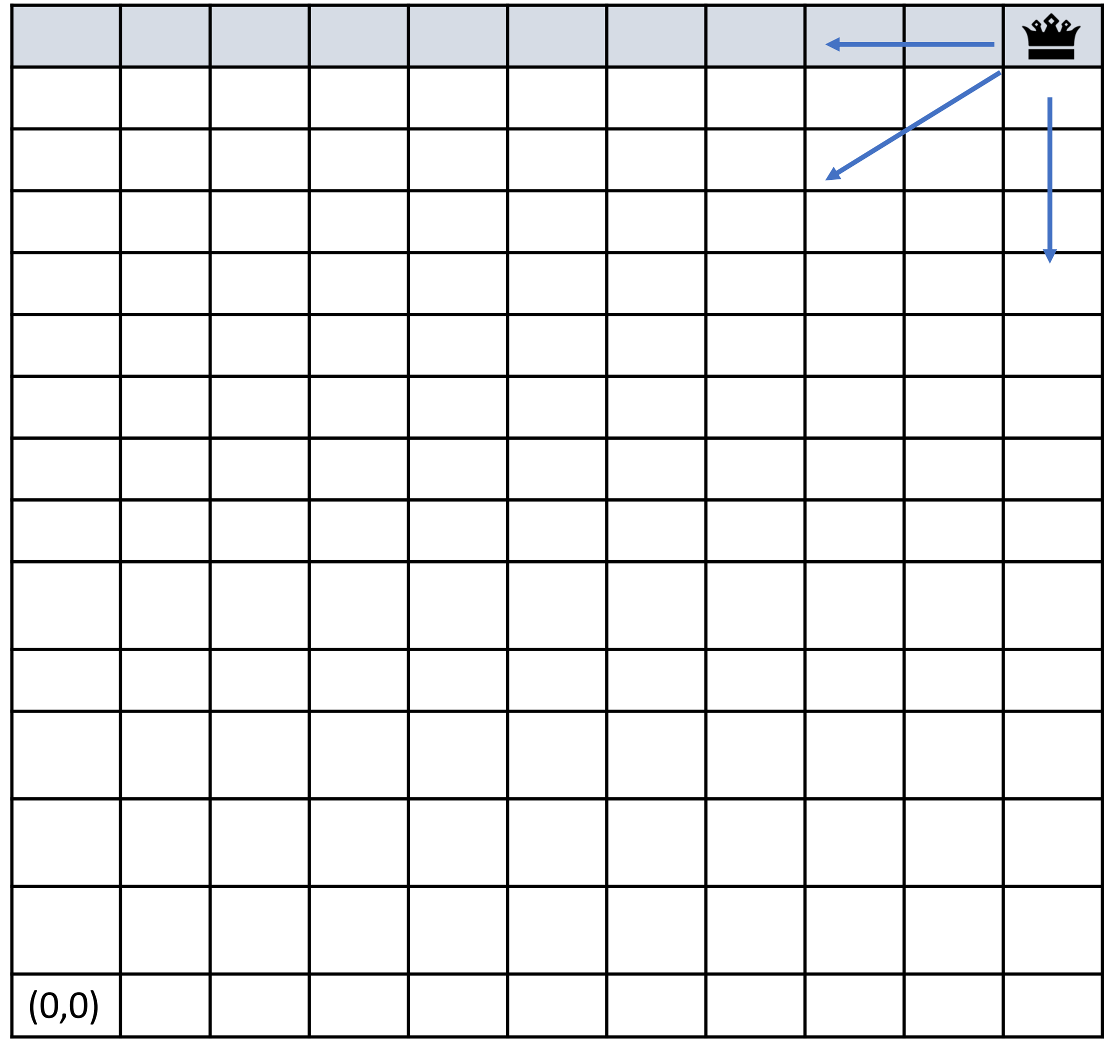
$(1,2)$, $(3,5)$, $(4,7)$, $(6,10)$, $(8,13)$, ...

$(1,10)$, $(100,1000)$, $(101,1010)$, $(1001,10010)$, $(10000,100000)$, ...

21	13	8	5	3	2	1

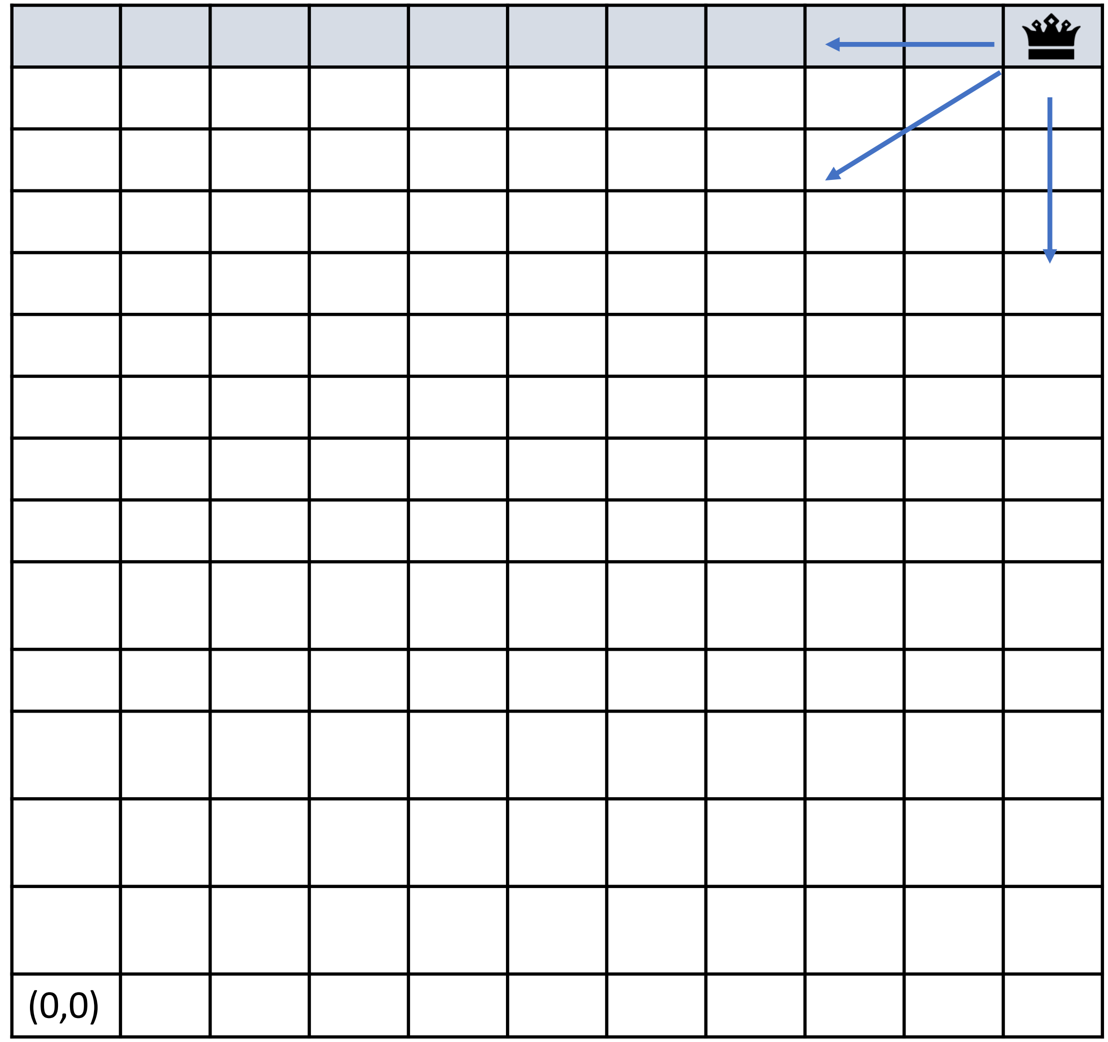
A rightmost 1 in even position

Is (10, 15) safe?



Is (10, 15) safe?

Write in terms of Fibonacci numbers



Is (10, 15) safe?

	21	13	8	5	3	2	1
10	0	0	1	0	0	1	0
15	0	1	0	0	0	1	0

Is (10, 15) safe?

	21	13	8	5	3	2	1
10	0	0	1	0	0	1	0
15	0	1	0	0	0	1	0

Are these an (A, B) pair?

Is (10, 15) safe?

Which row do we take from?

	21	13	8	5	3	2	1
10	0	0	1	0	0	1	0
15	0	1	0	0	0	1	0

Is (10, 15) safe?

Let's say the second

	21	13	8	5	3	2	1
10	0	0	1	0	0	1	0
15	0	1	0	0	0	1	0

Is (10, 15) safe?

	21	13	8	5	3	2	1
10	0	0	1	0	0	1	0
15	0	1	0	0	0	1	0

Let's say the second

Can we make $(10001, 100010)$?

Is (10, 15) safe?

	21	13	8	5	3	2	1
10	0	0	1	0	0	1	0
15	0	1	0	0	0	1	0

Let's say the second

Can we make $(10001, 100010)$?

Nope, 10001 is 14

Is (10, 15) safe?

	21	13	8	5	3	2	1
10	0	0	1	0	0	1	0
15	0	1	0	0	0	1	0

How about the first?

Is (10, 15) safe?

	21	13	8	5	3	2	1
10	0	0	1	0	0	1	0
15	0	1	0	0	0	1	0

How about the first?

Can we make $(10010, 100100)$?

Is (10, 15) safe?

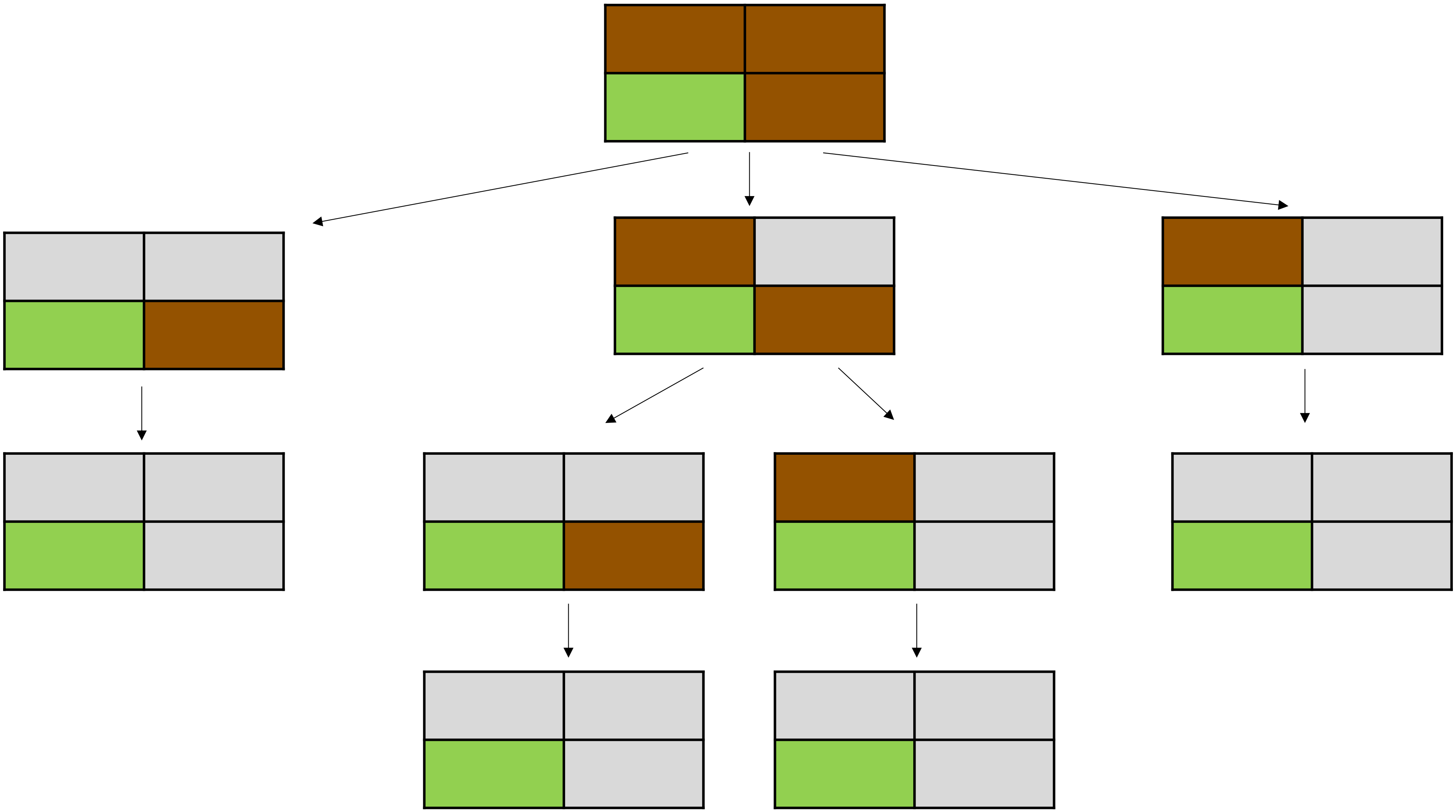
	21	13	8	5	3	2	1
10	0	0	1	0	0	1	0
15	0	1	0	0	0	1	0

How about the second?

Can we make $(10010, 100100)$?

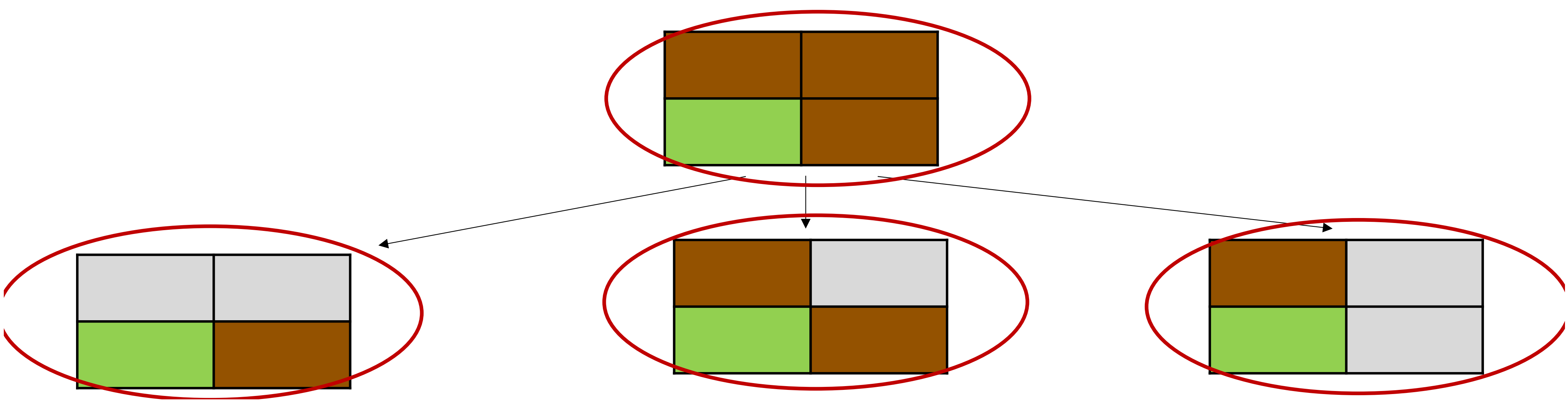
100100 is 9 so move a step left

Recap



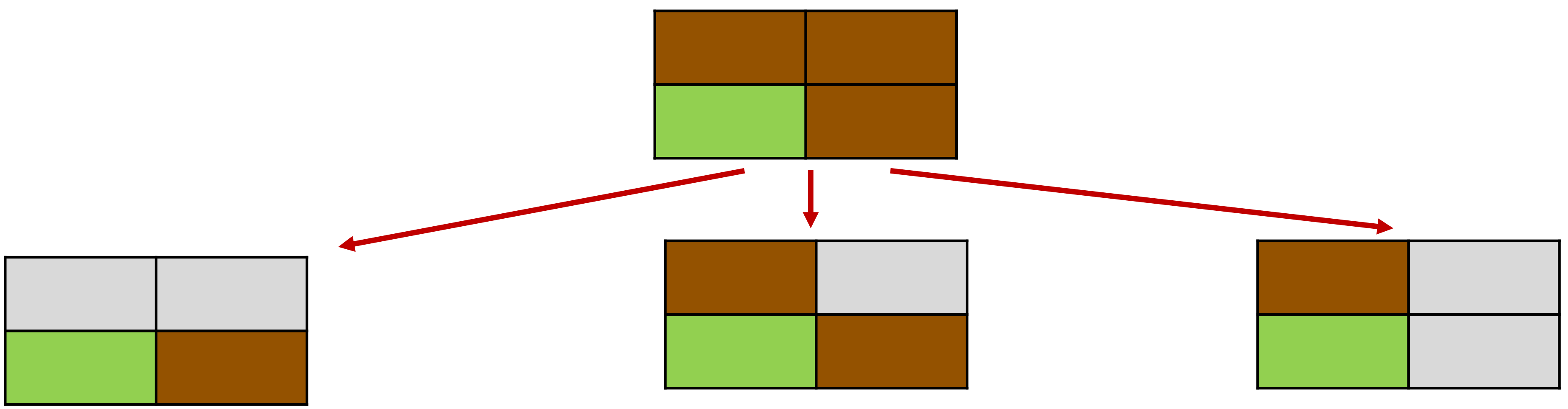
A game

A set of positions each allowing a set of moves



A game

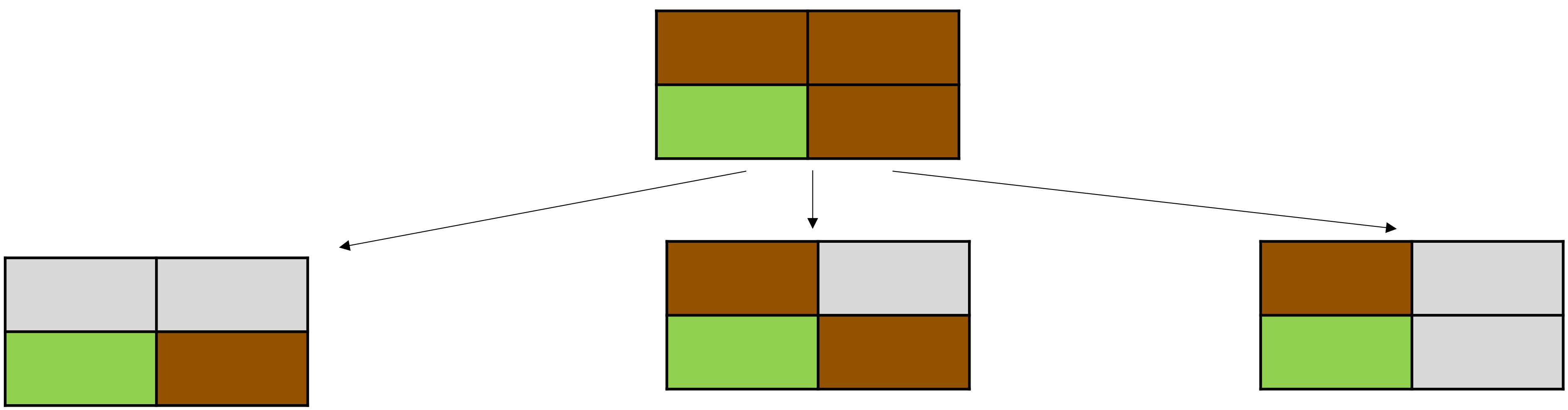
A set of positions each allowing a set of moves



A game

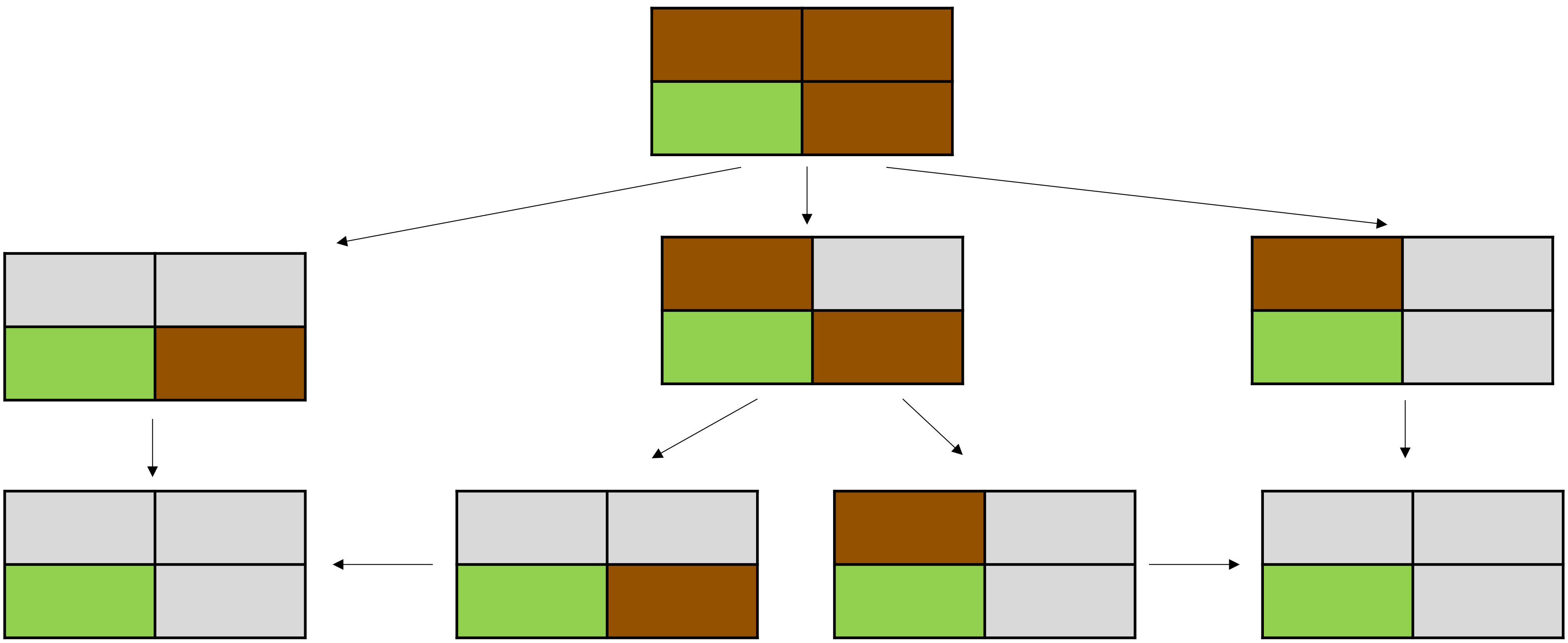
A set of positions each allowing a set of moves

Game ends if the current player cannot move



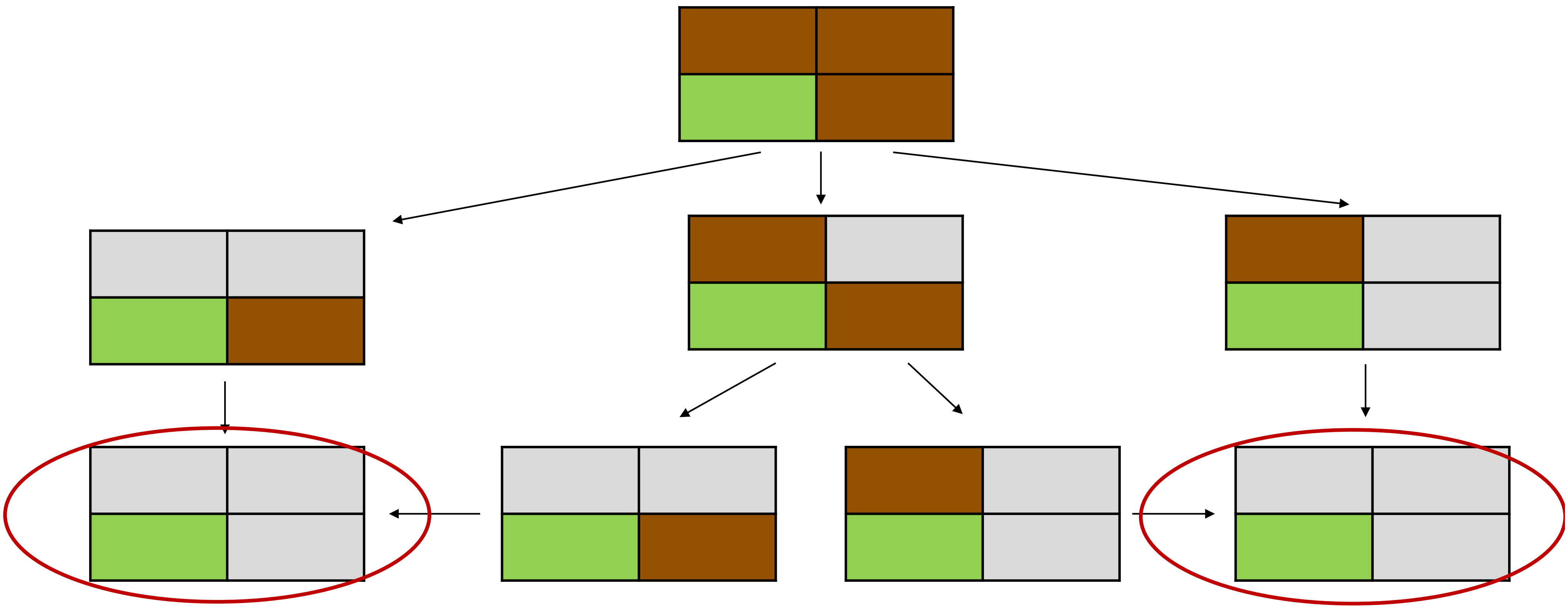
N and P positions

Terminal positions are P



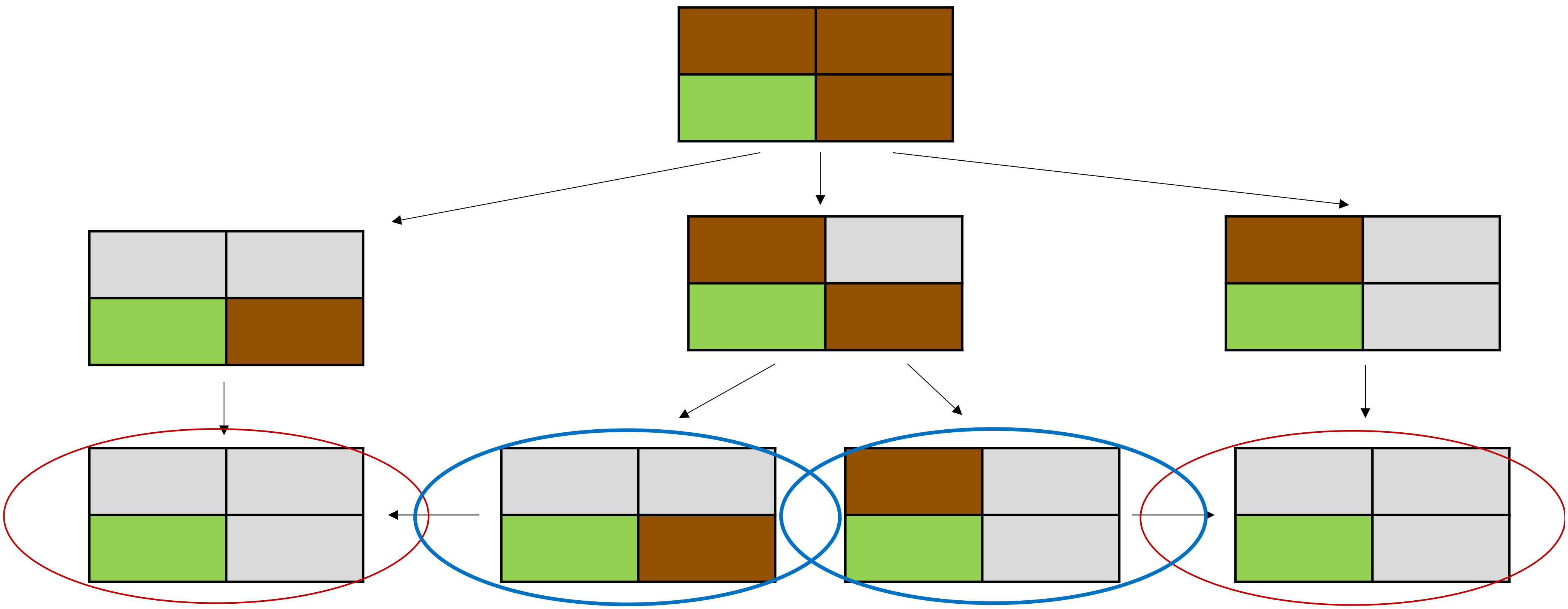
N and P positions

Terminal positions are P



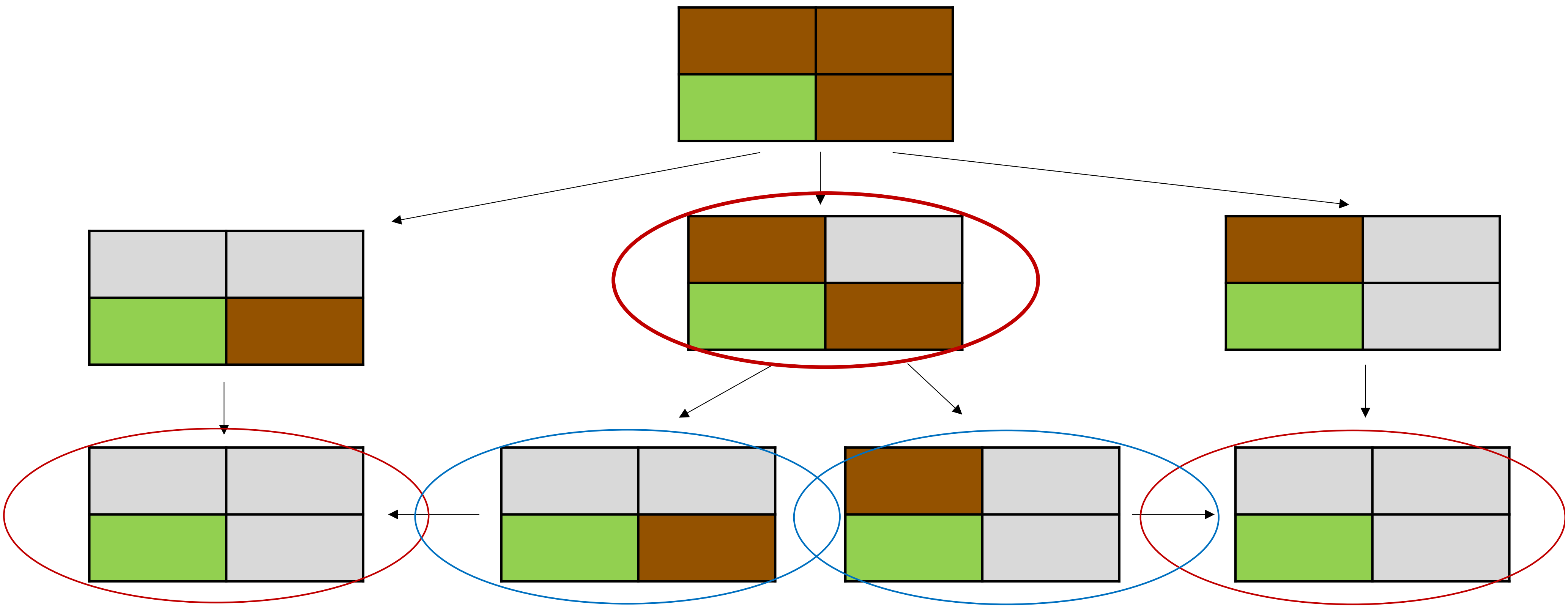
N and P positions

N if it has a P child



N and P positions

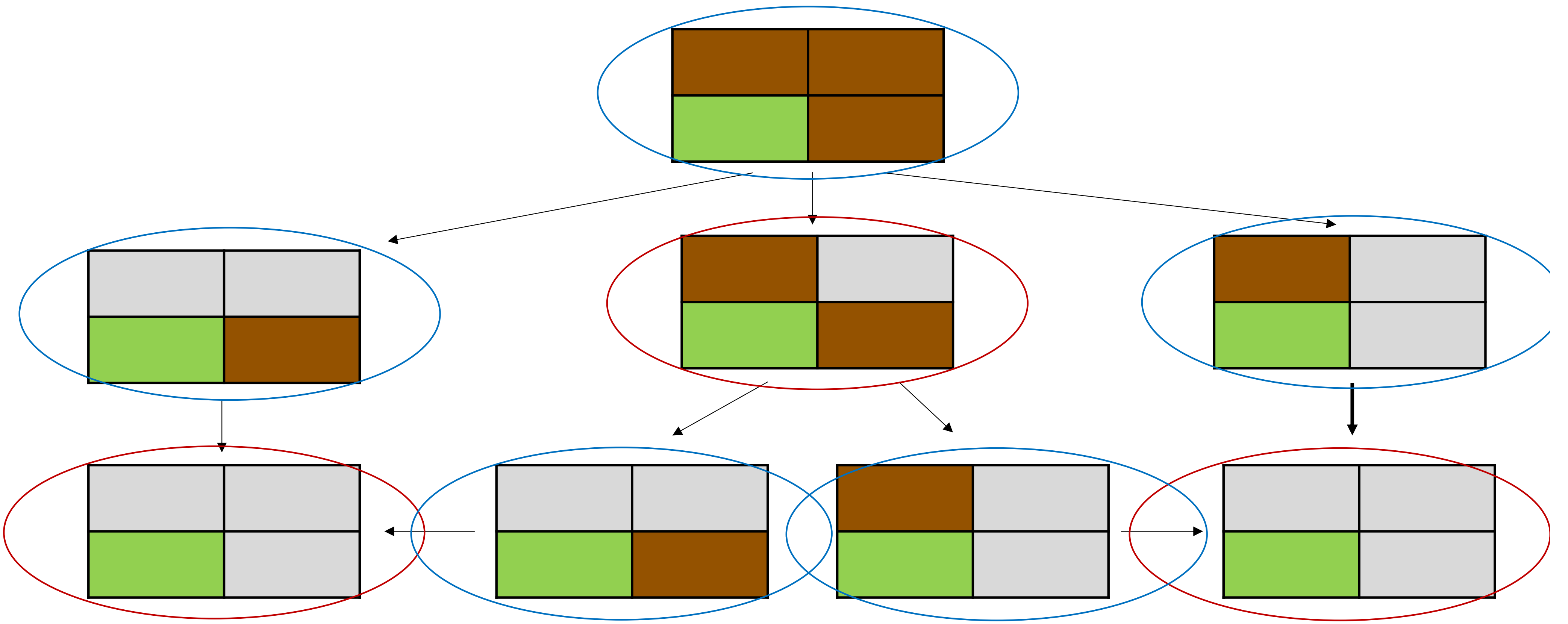
P if all children are N



N and P positions

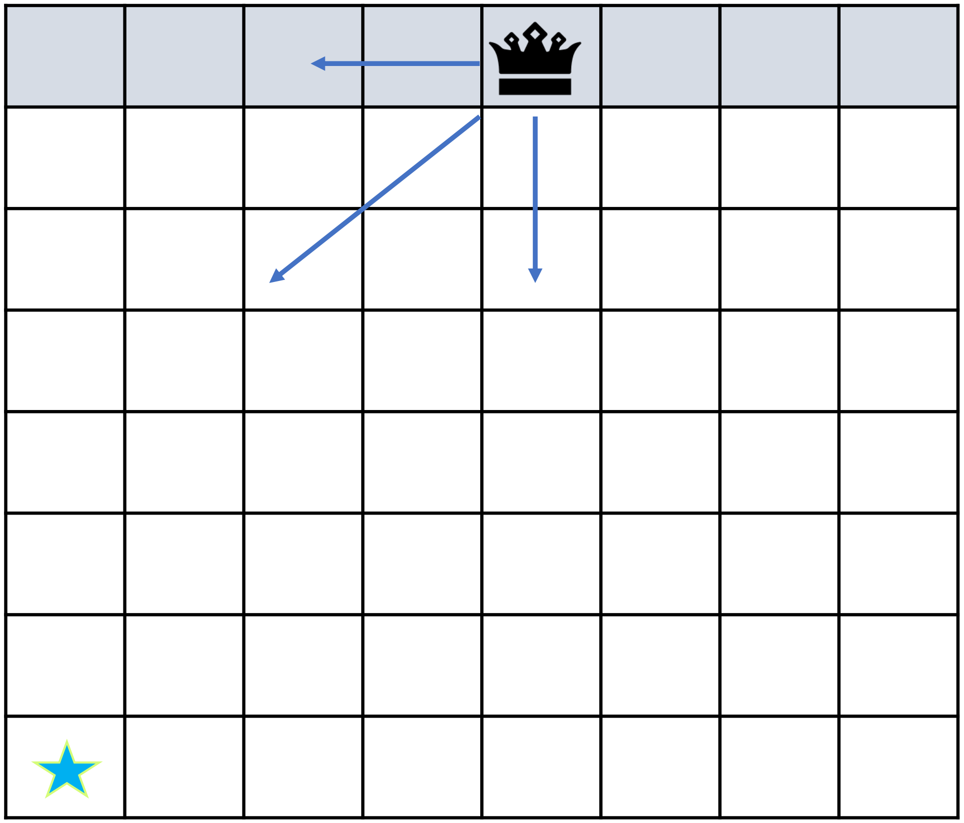
P if all children are N

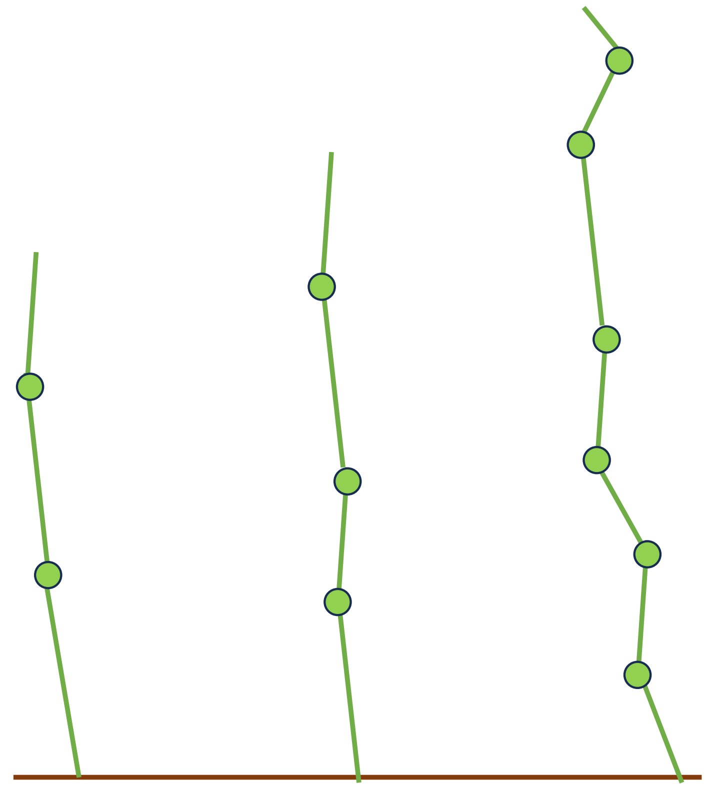
N if it has a P child



Let's Play!

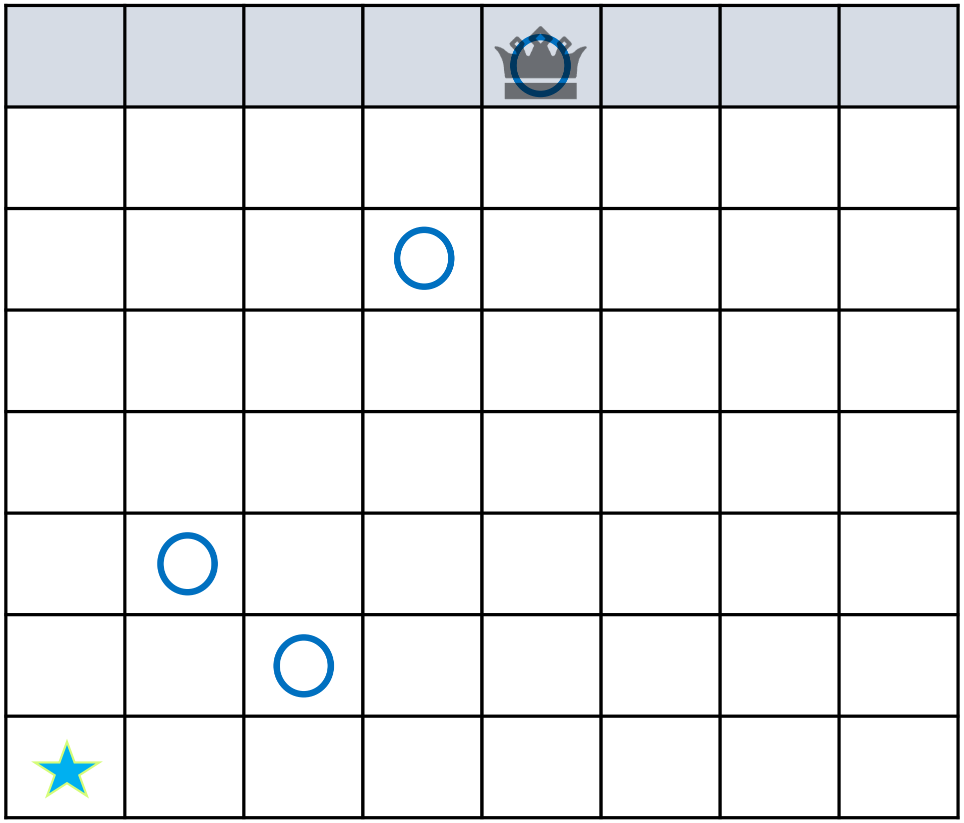
Simultaneously play

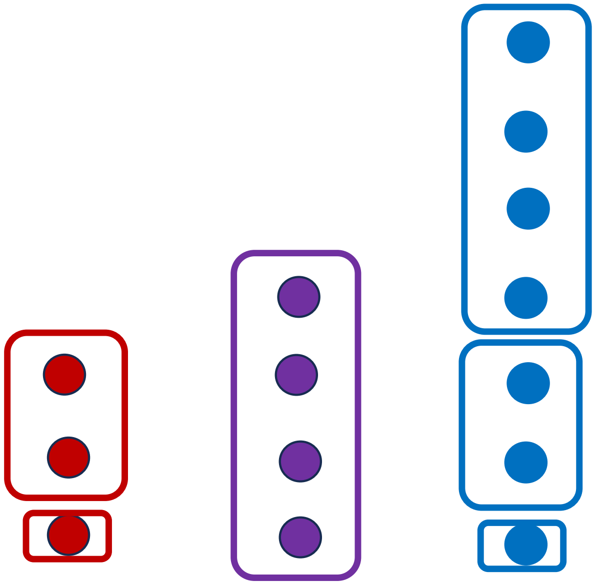




Simultaneously play

Both are P positions

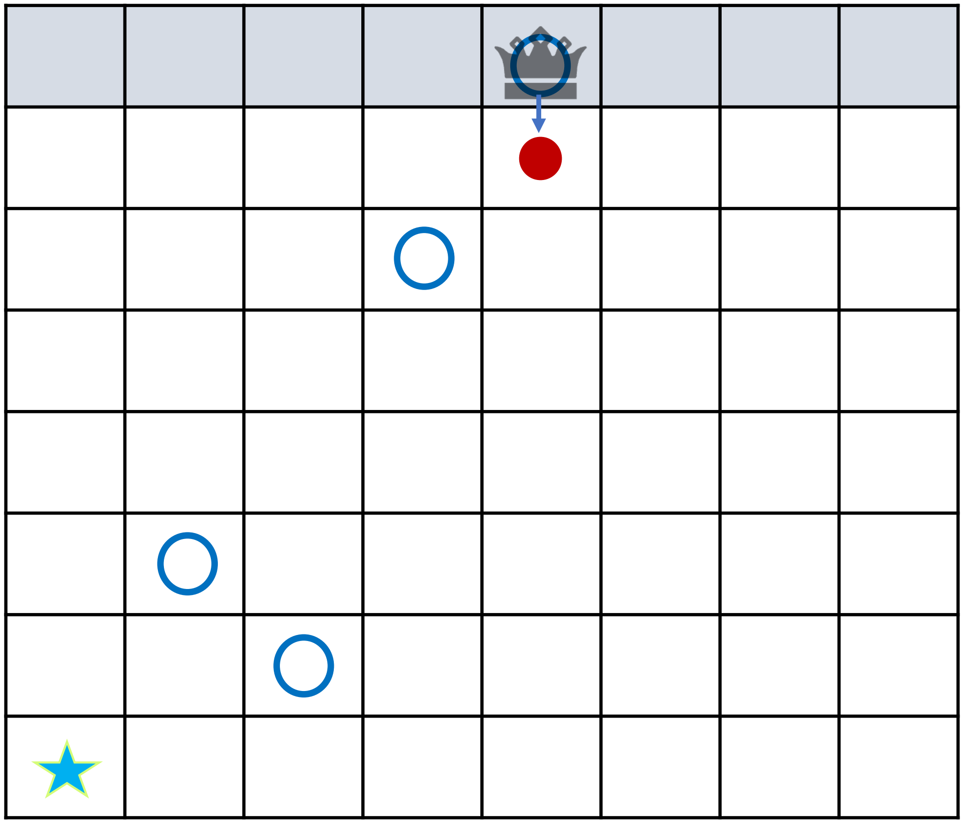


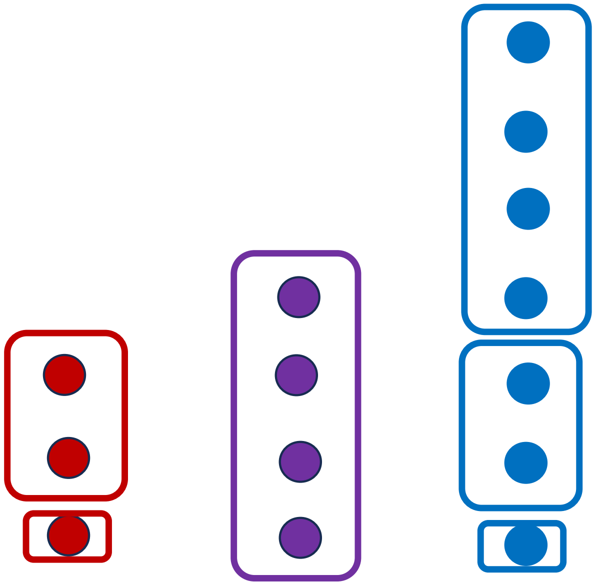


Simultaneously play

Both are P positions

Any move is to N



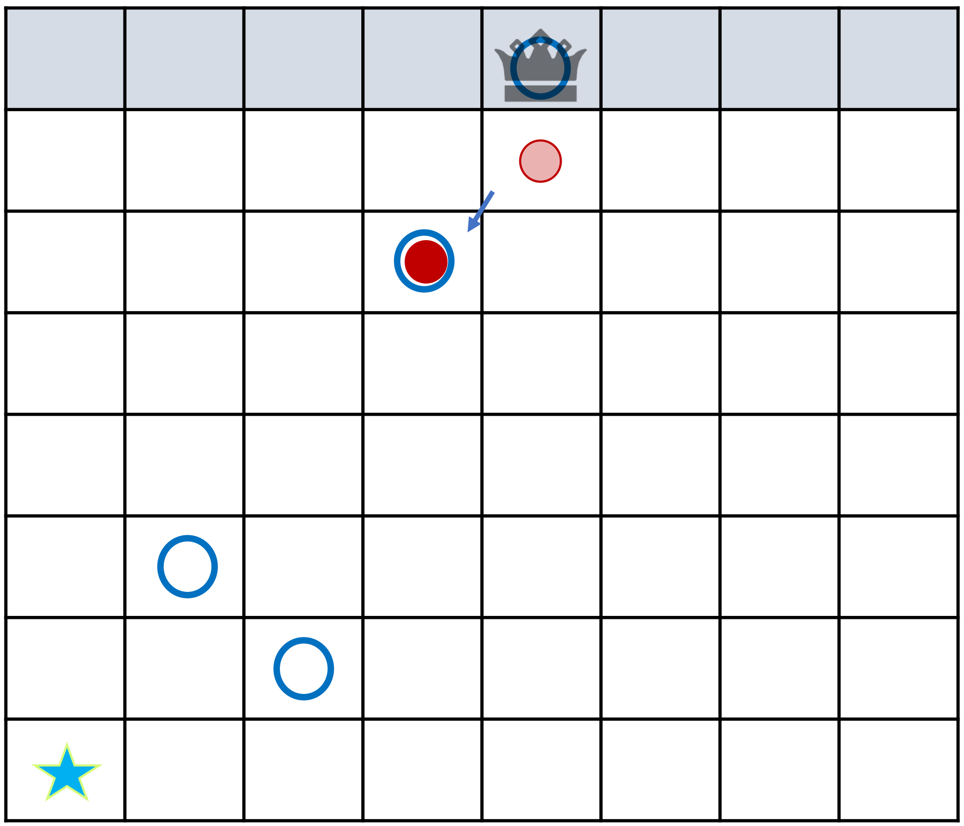


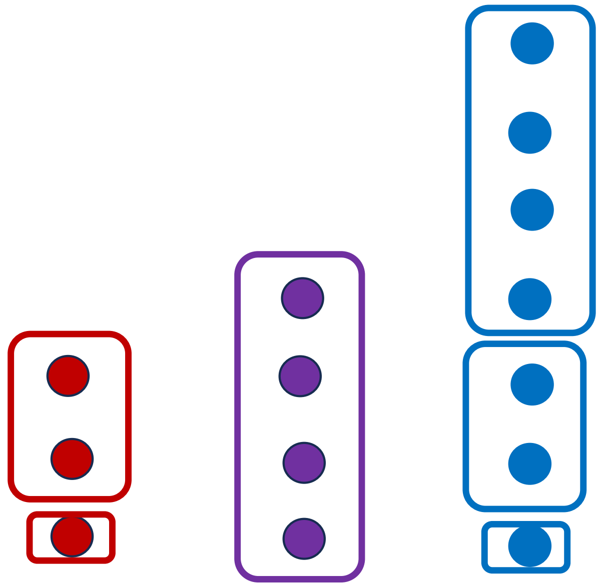
Simultaneously play

Both are P positions

Any move is to N

Make a move in same game to restore Property



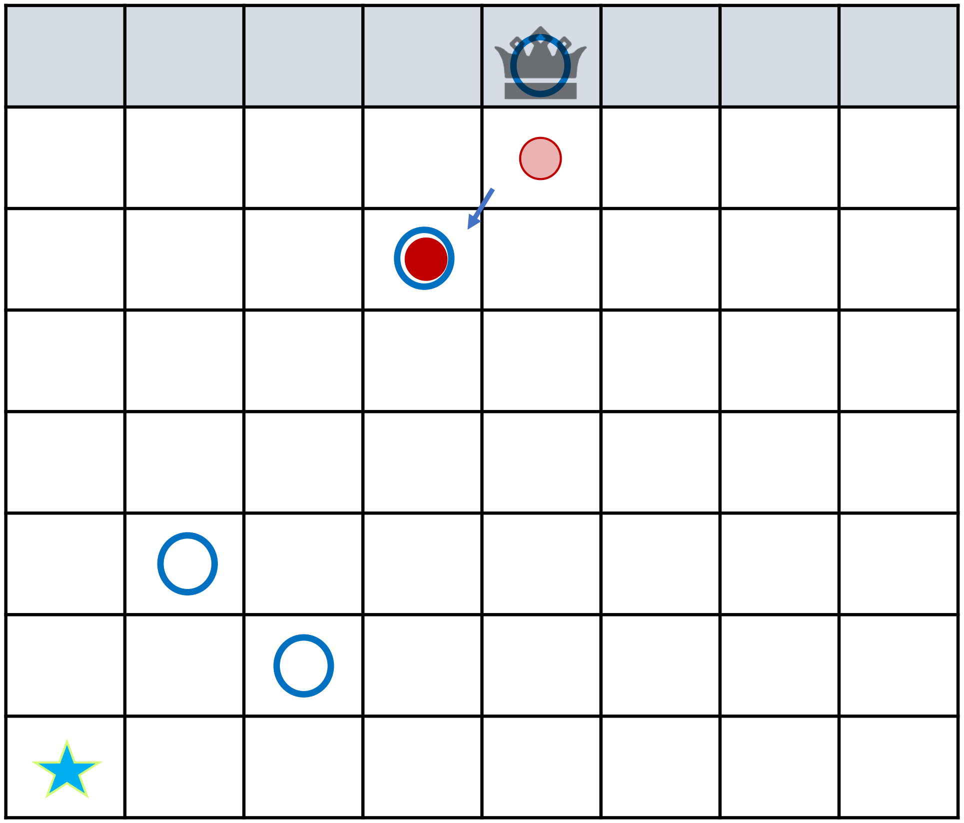


Simultaneously play

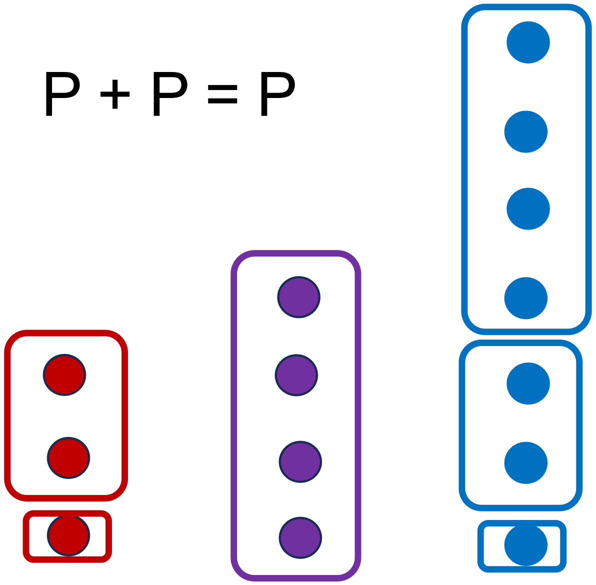
Both are P positions

Any move is to N

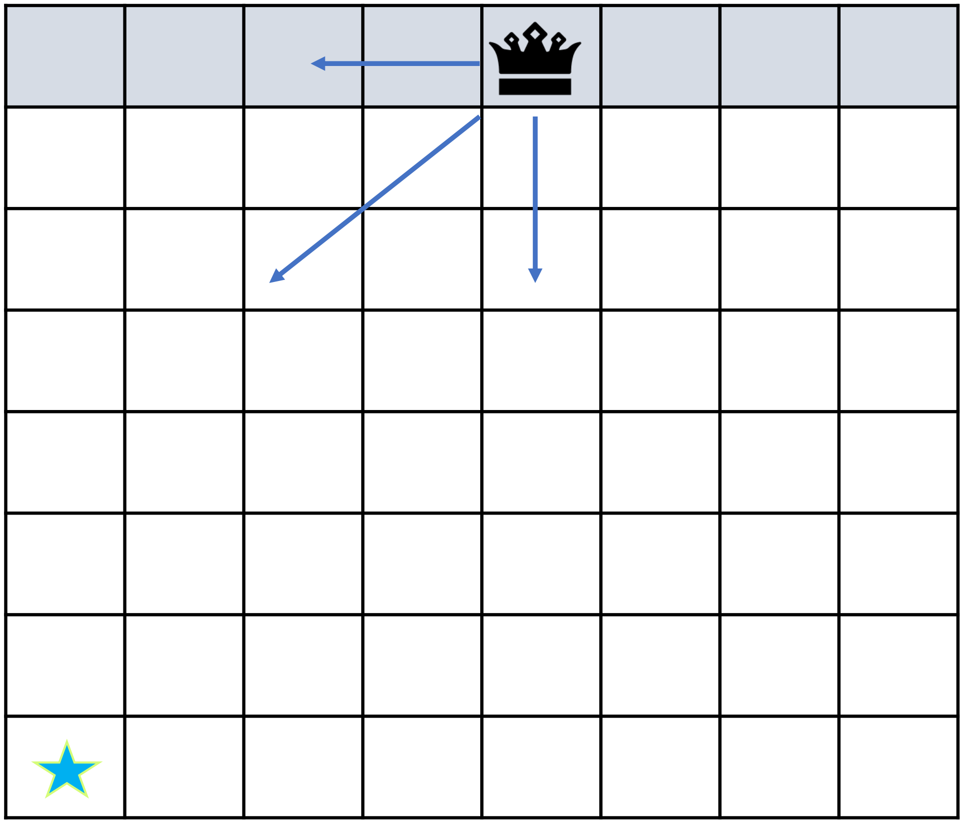
Make a move in same game to restore Property

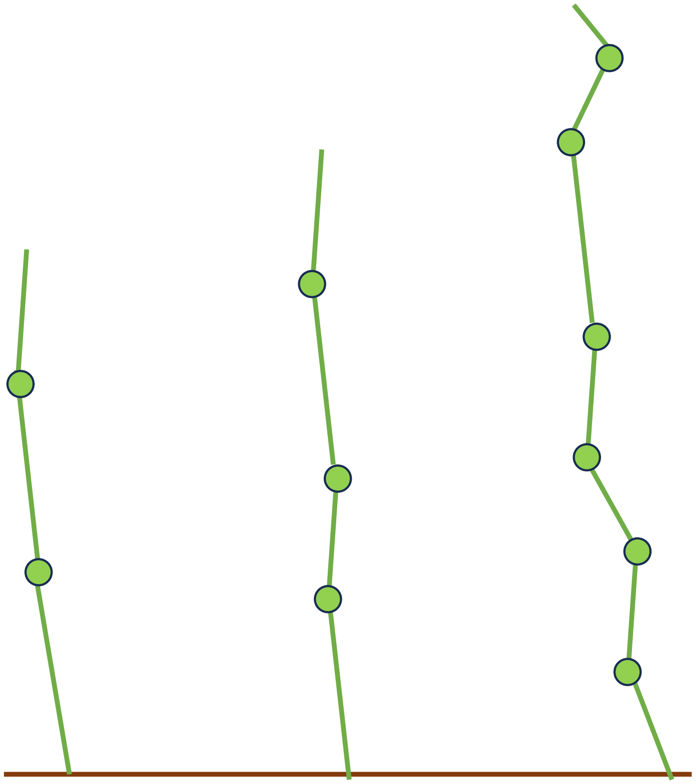


P + P = P



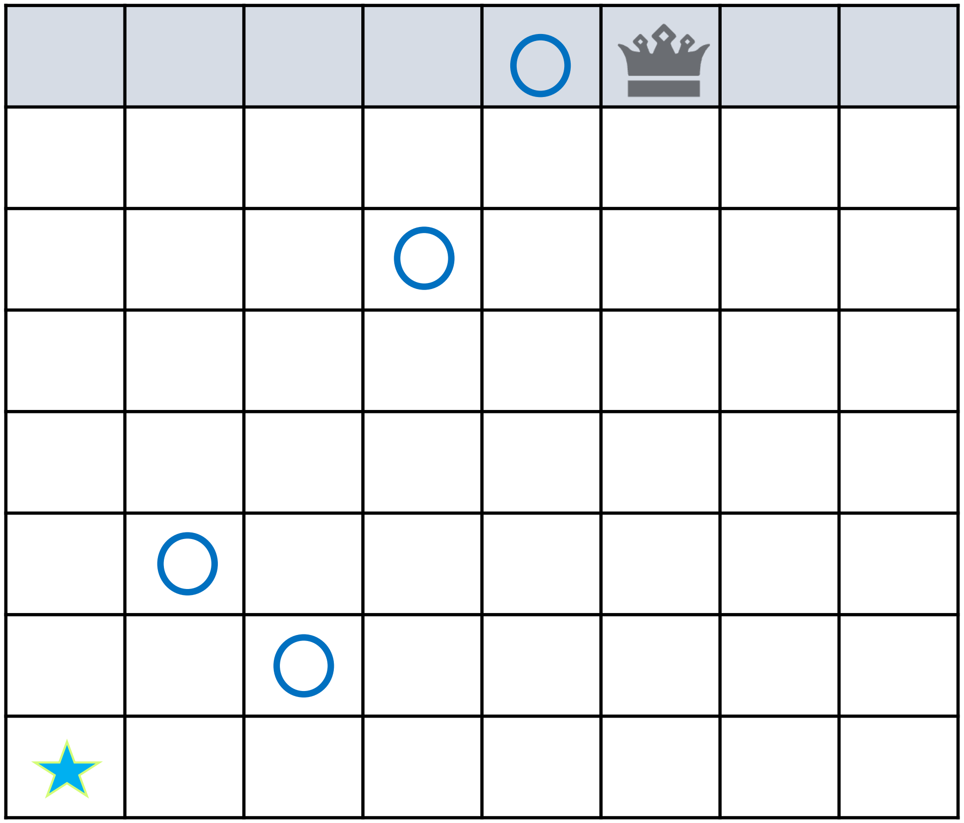
Simultaneously play

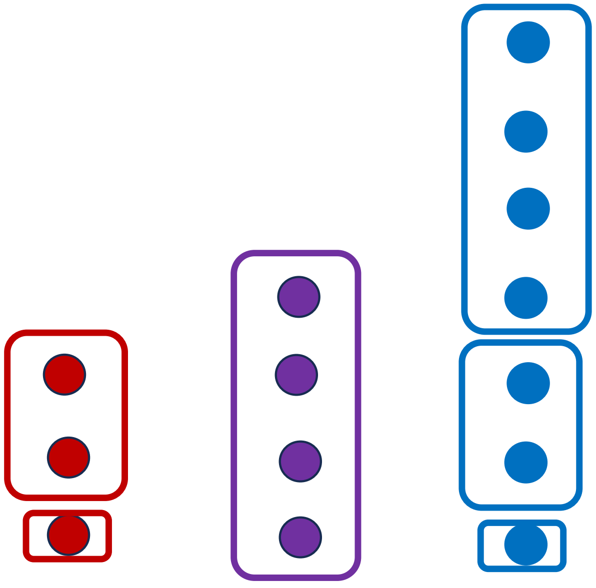




Simultaneously play

One N and one P position

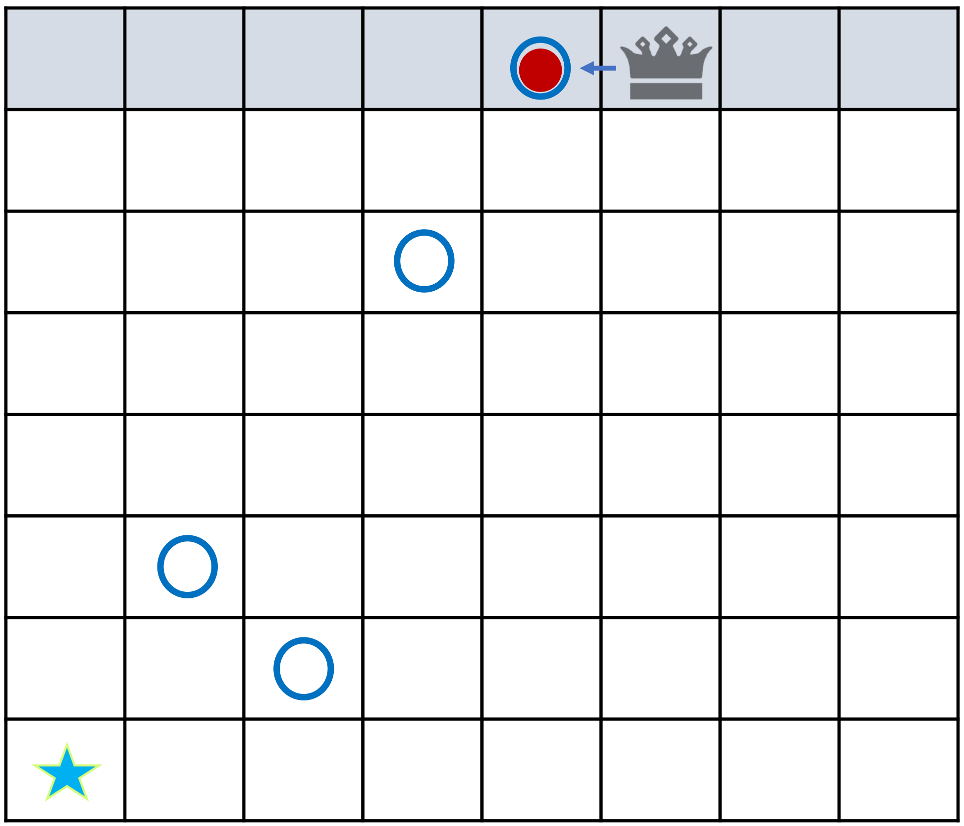


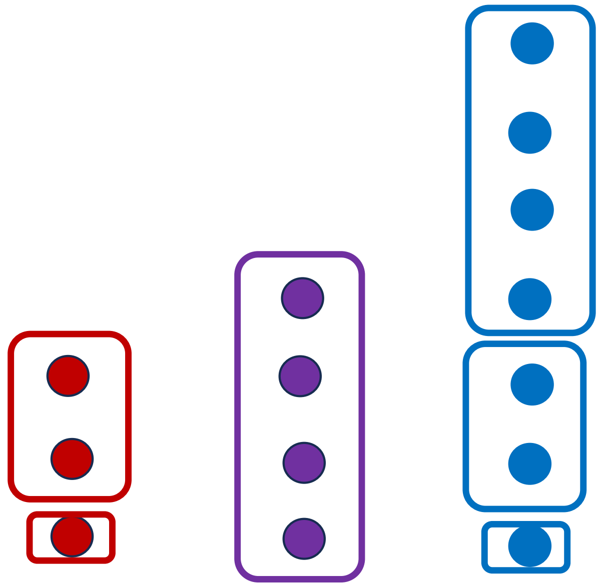


Simultaneously play

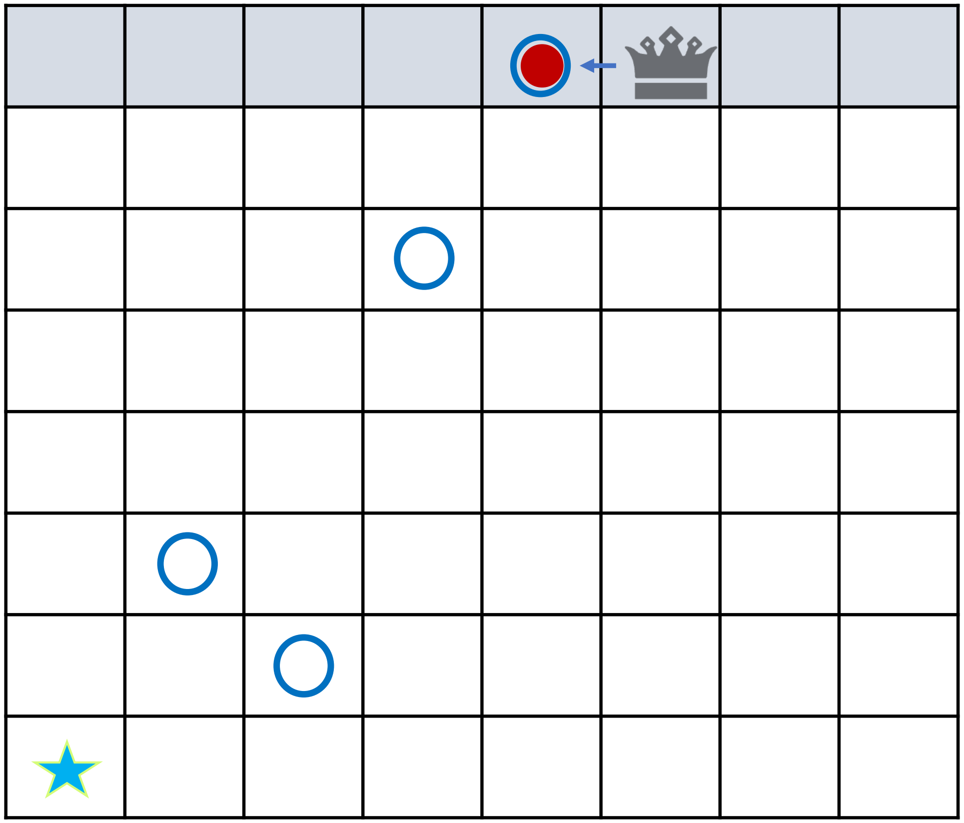
One N and one P position

Move the N game to a P position





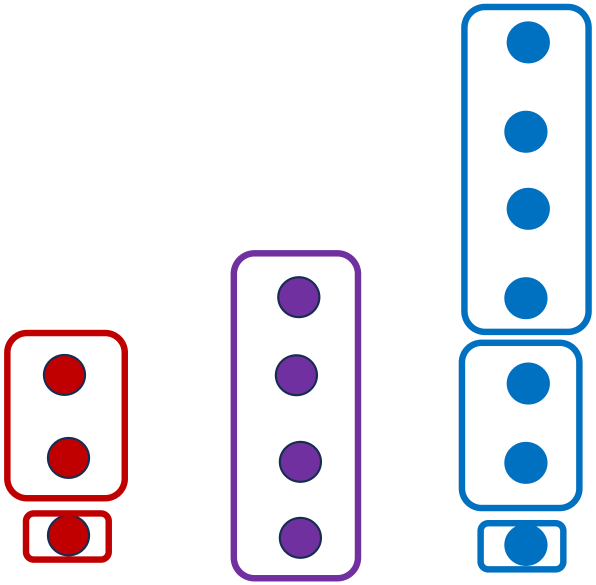
Simultaneously play



Both are P positions

Move the N game to a P position

Now both are P positions

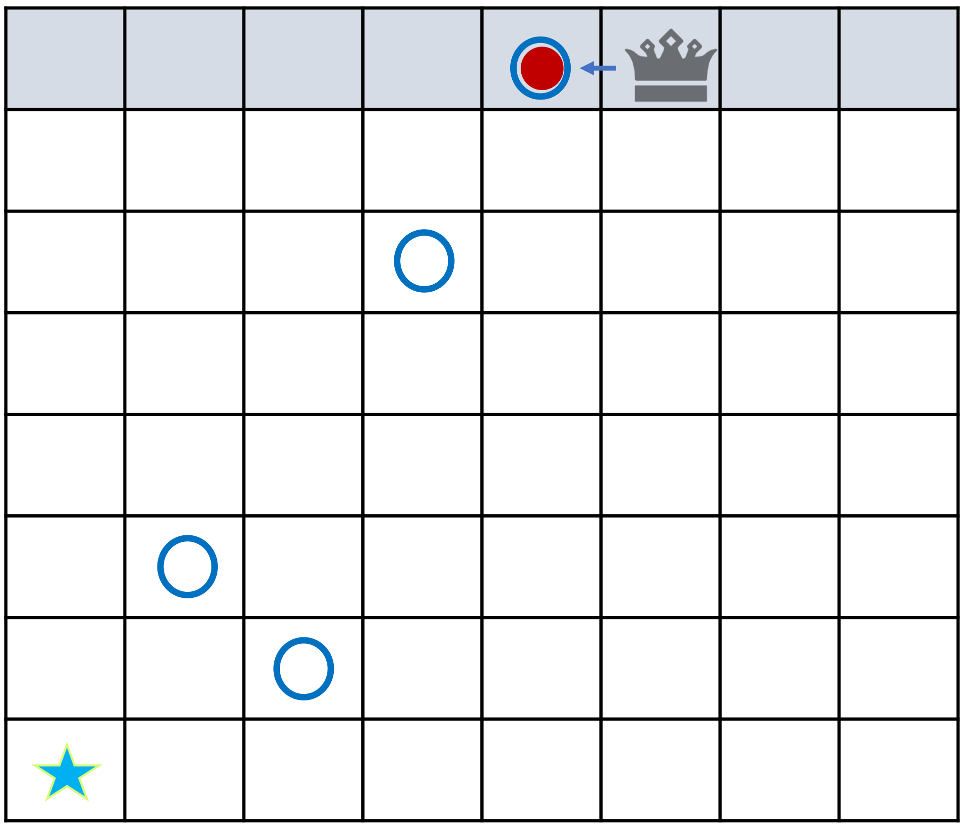


Simultaneously play

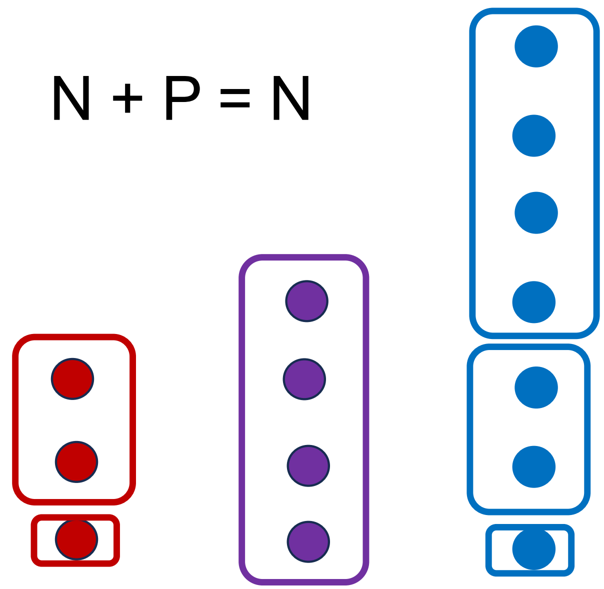
Both are P positions

Move the N game to a P position

Now both are P positions



N + P = N



Equivalent games

Two Games, $G, H,$ are equivalent if

$$\boldsymbol{o(G+K)=o(H+K)}$$

for all games K

where $o()$ is the outcome class of the game

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$$o(G+K) = o(H+K)$$

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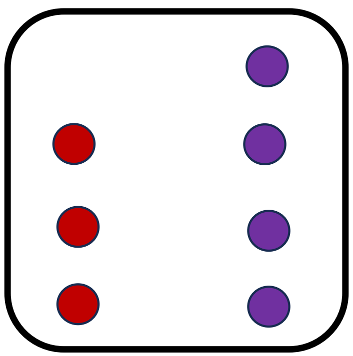
Sprague-Grundy Theorem

Any finite impartial game is equivalent to a single Nim heap

$$G = * n$$

Sprague-Grundy Theorem

Any finite impartial game is equivalent to a single Nim heap

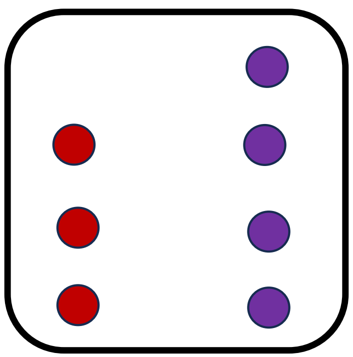


G

$$G = * n$$

Sprague-Grundy Theorem

Any finite impartial game is equivalent to a single Nim heap



G

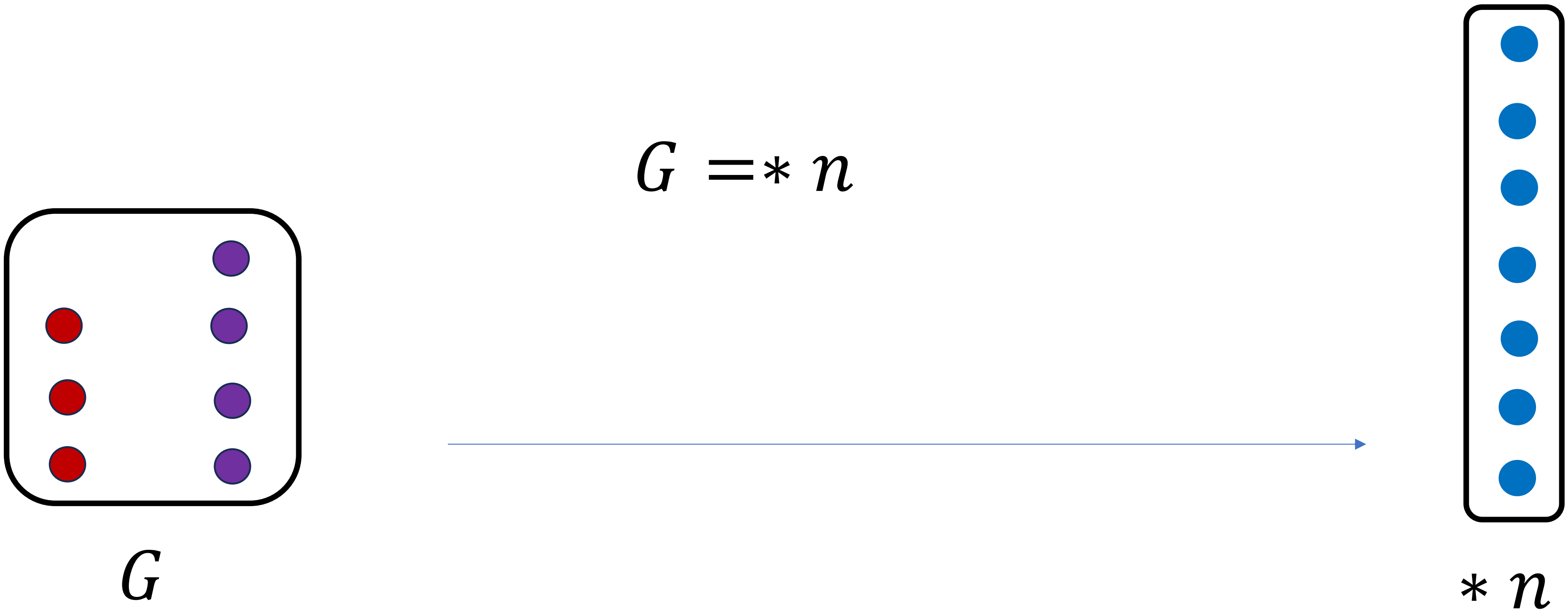
$$G =* n$$

$$3 \oplus 4 \quad \begin{array}{c} 011 \\ 100 \\ \text{-------} \\ 111 \end{array} \quad n = 7$$

Sprague-Grundy Theorem

Any finite impartial game is equivalent to a single Nim heap

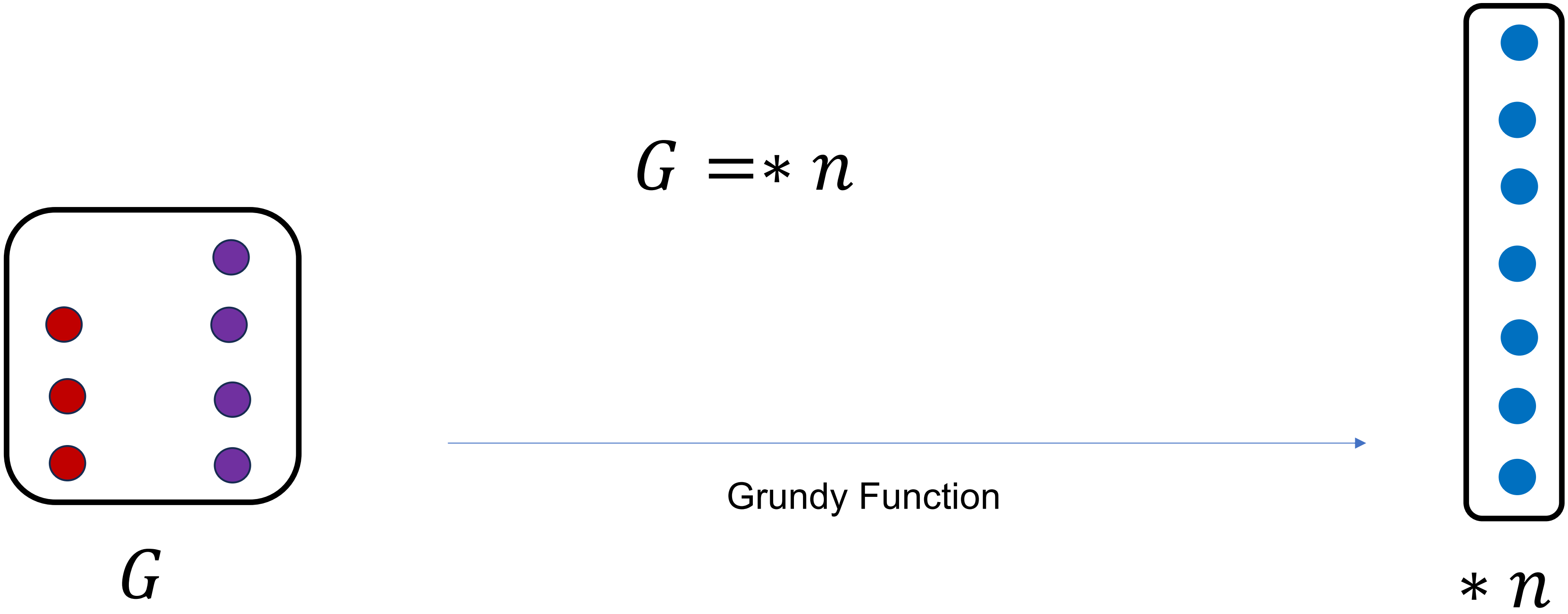
$$G = *n$$



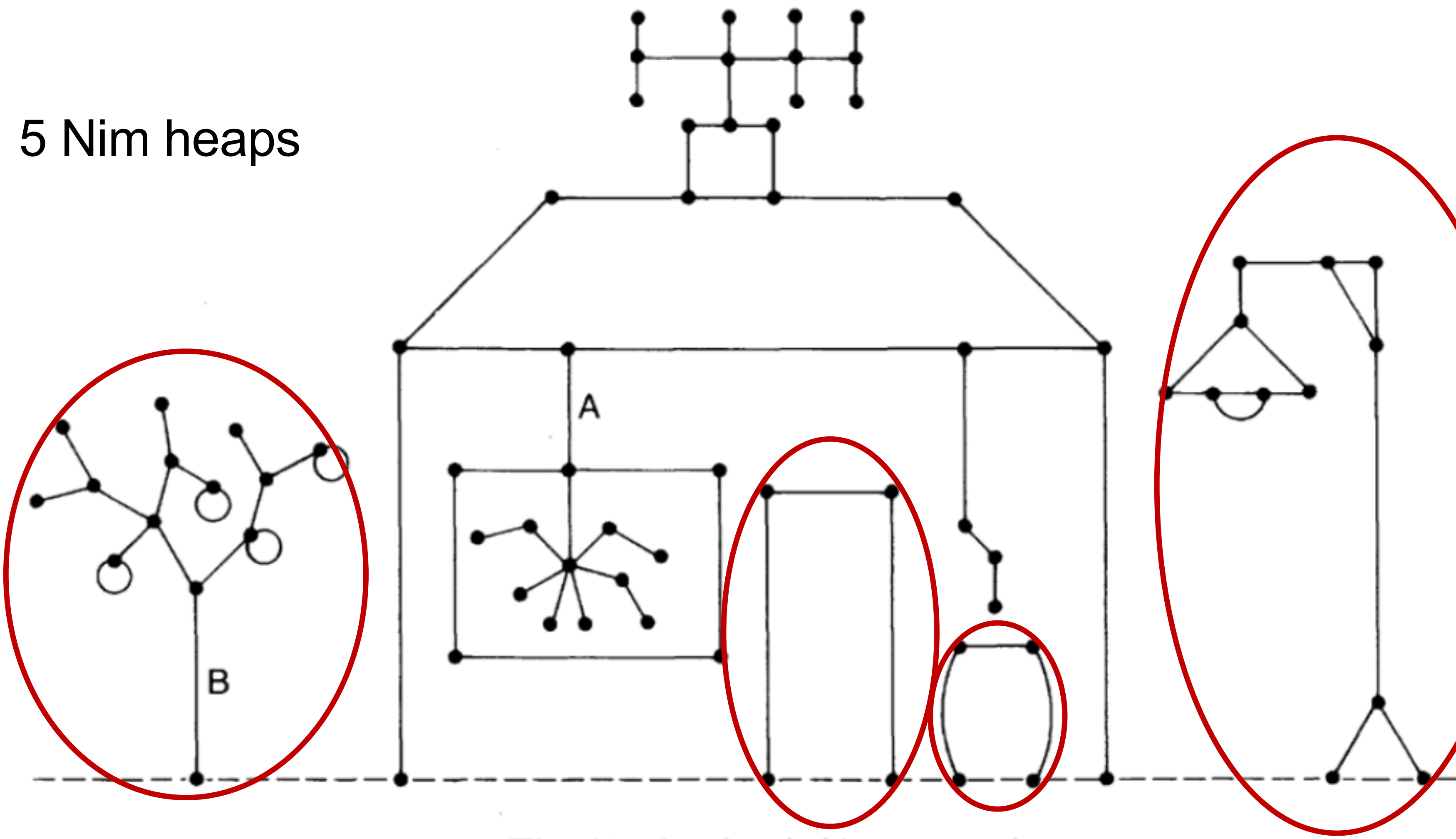
Sprague-Grundy Theorem

Any finite impartial game is equivalent to a single Nim heap

$$G =* n$$

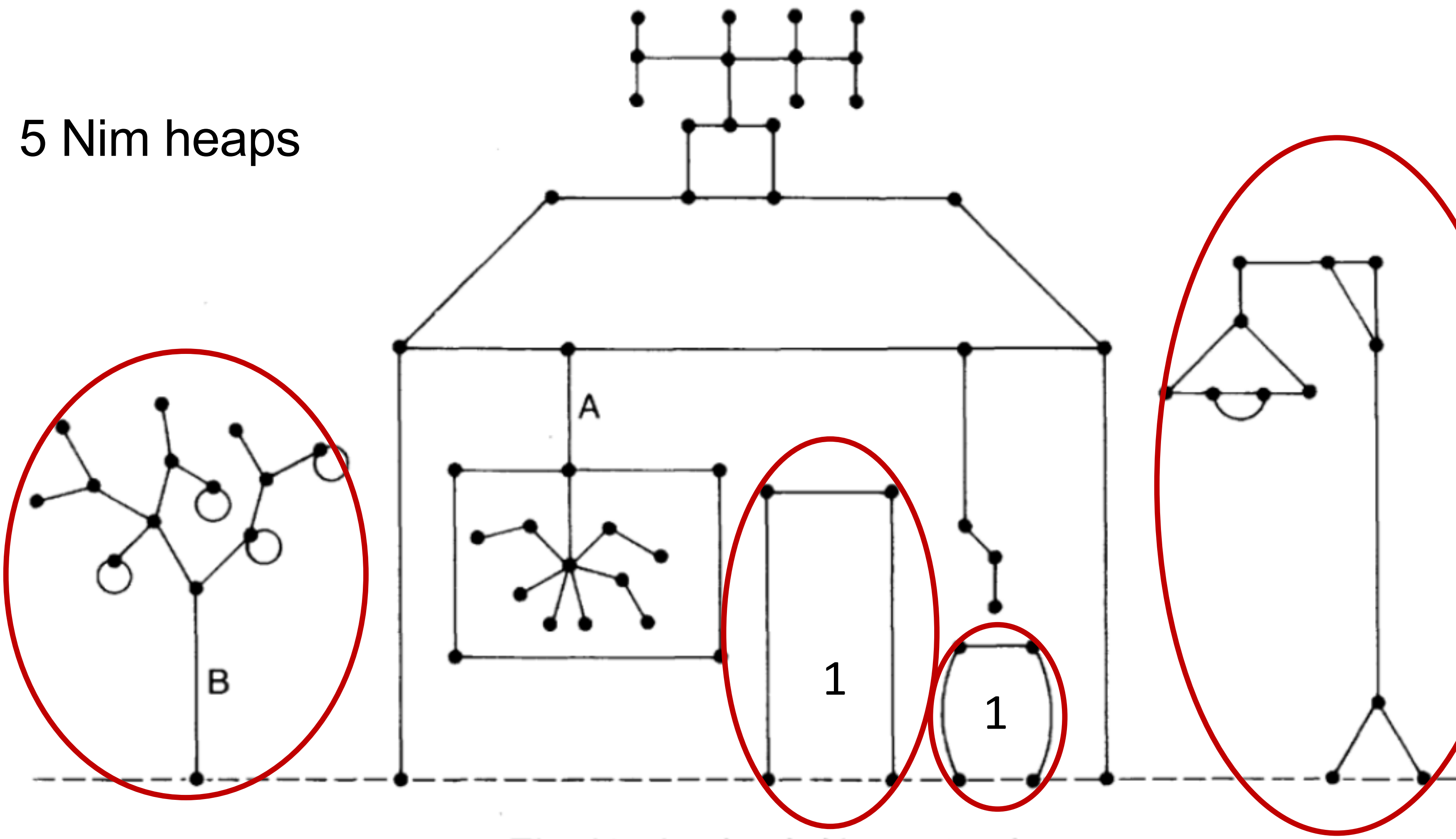


5 Nim heaps



The Hackenbush Homestead

5 Nim heaps

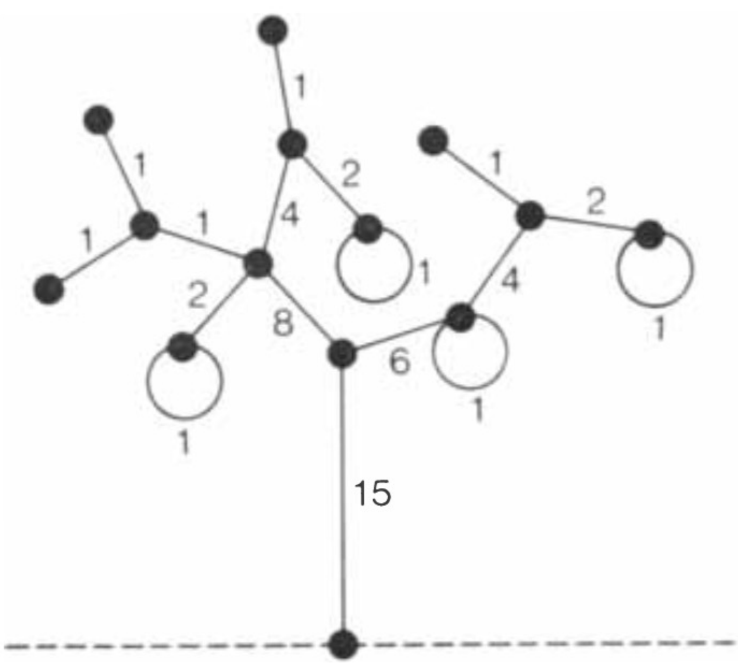


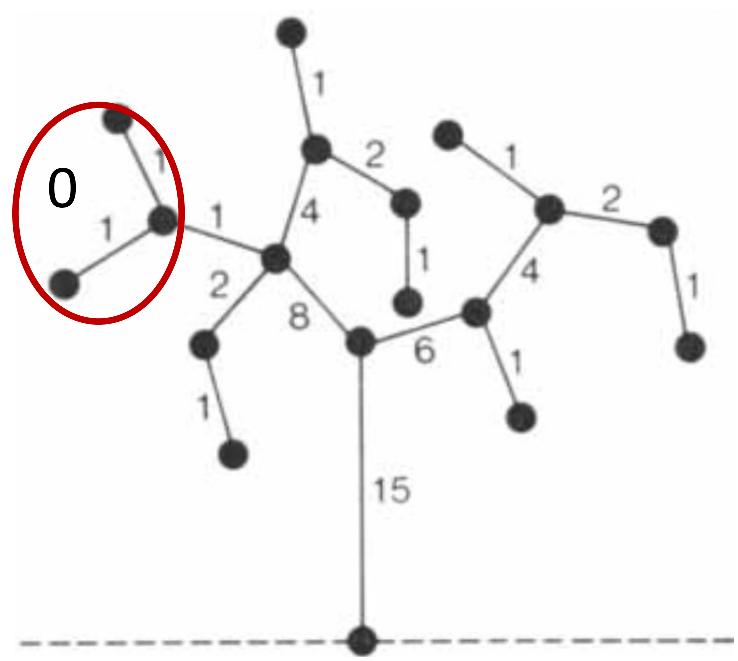
The Hackenbush Homestead

Apple Tree

$1 \oplus 1$

$$\begin{array}{r} 001 \\ 001 \\ \hline 000 \end{array}$$

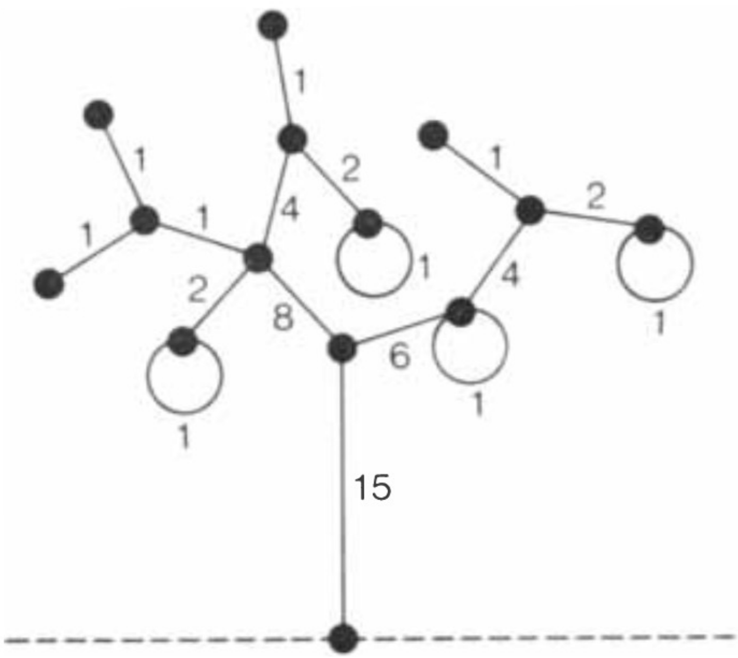


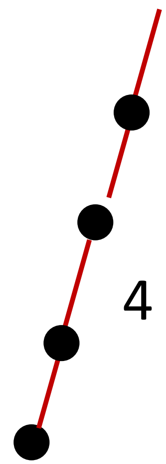


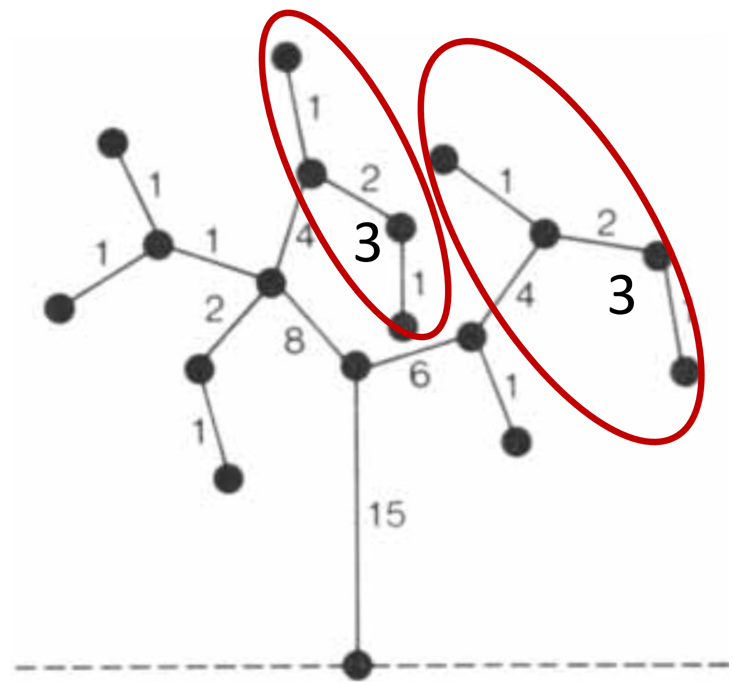
Apple Tree

$2 \oplus 1$

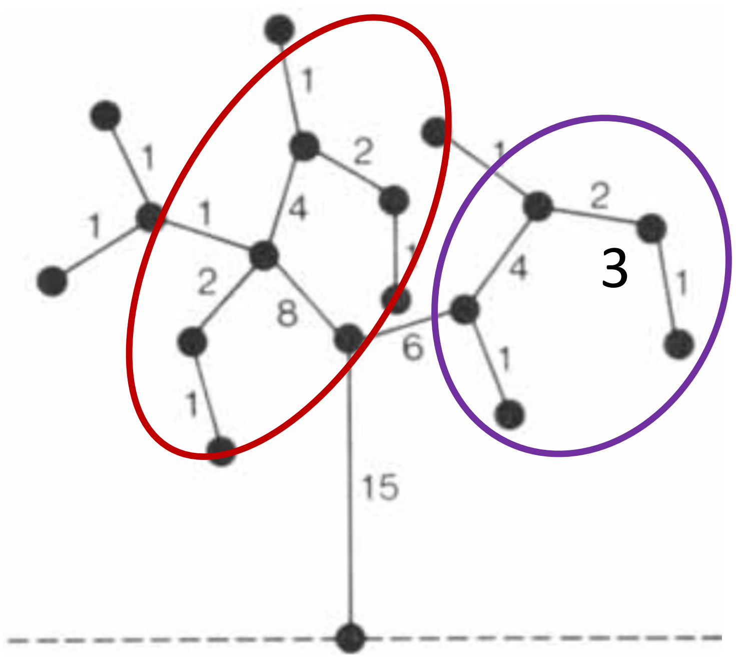
$$\begin{array}{r} 010 \\ 001 \\ \hline 011 \end{array}$$







Apple Tree



2⊕1⊕4

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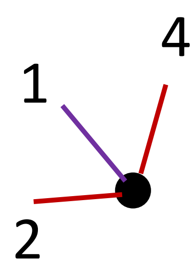
001
100

111



4⊕1

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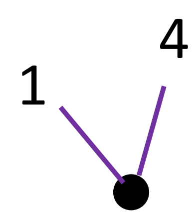
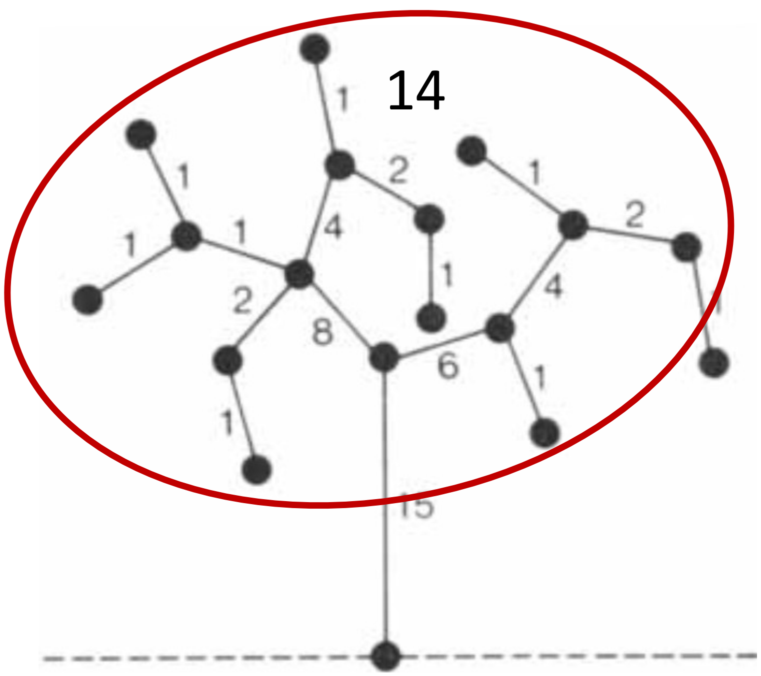
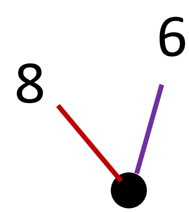


Apple Tree

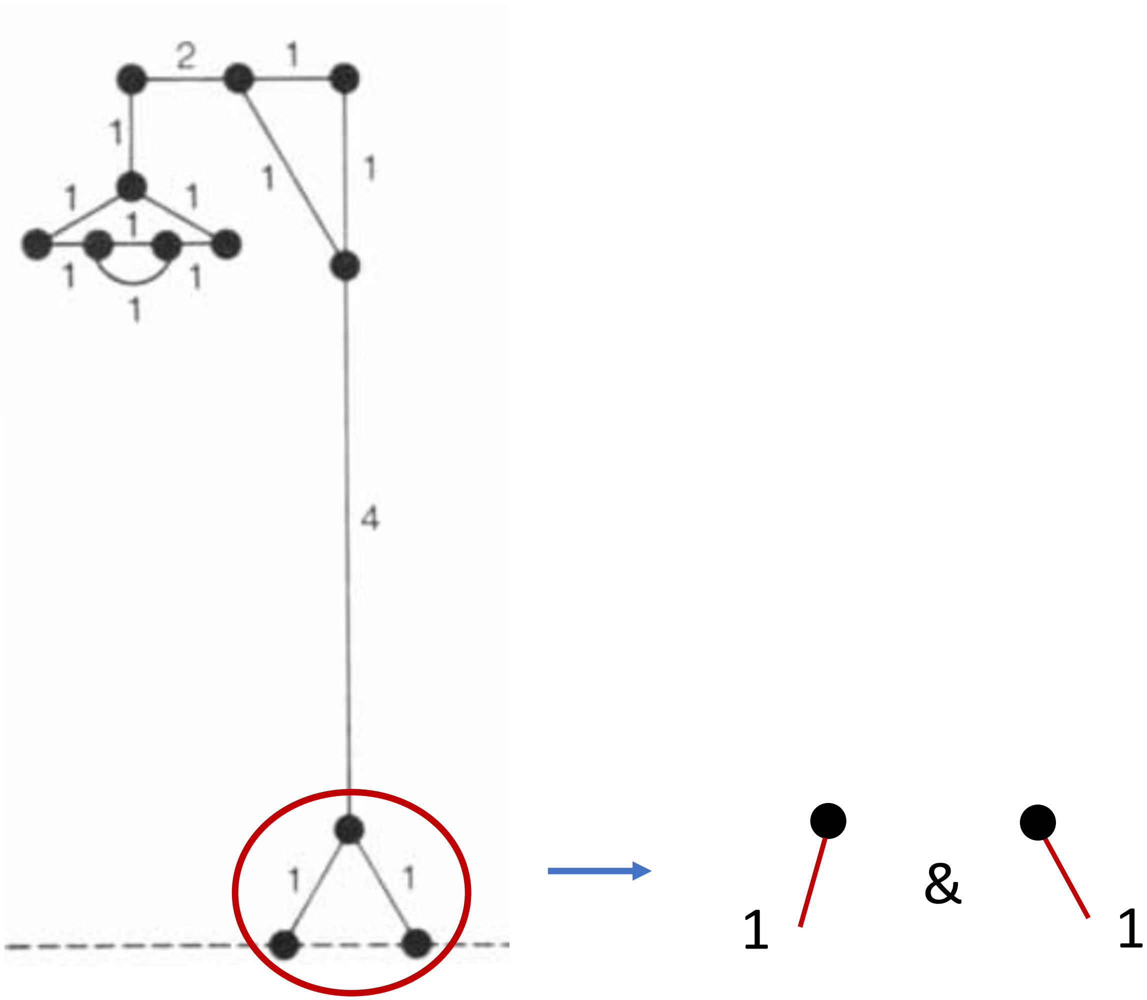




$8 \oplus 6$

1000
0110

1110


2
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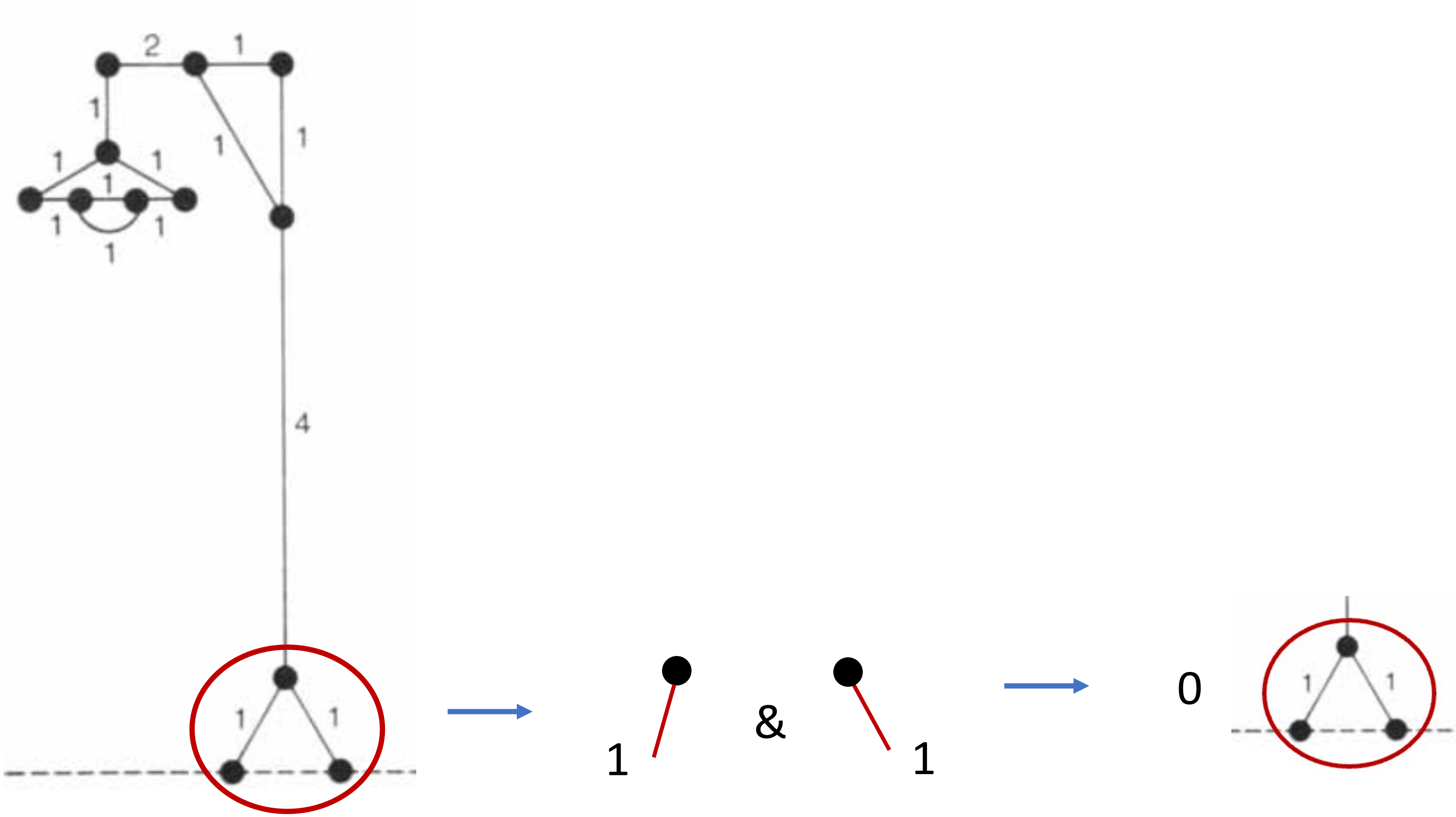

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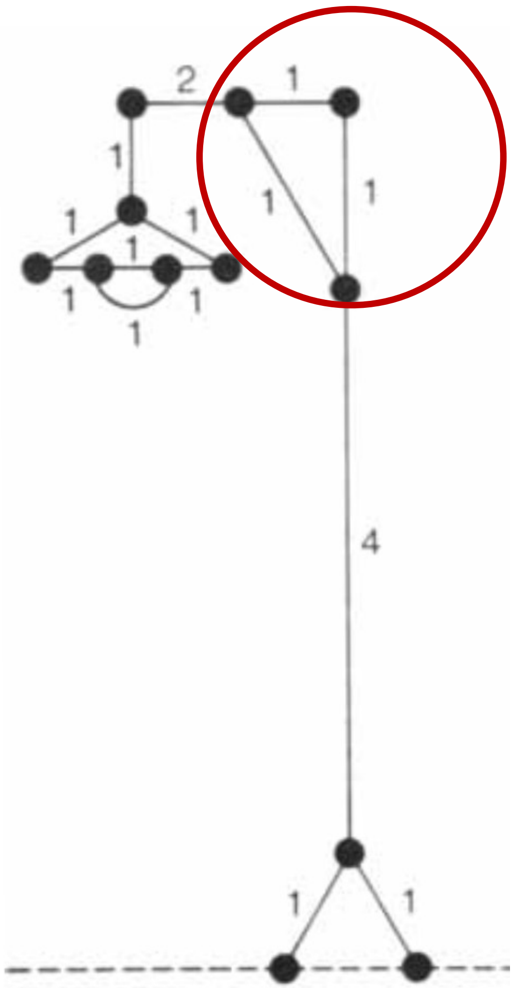

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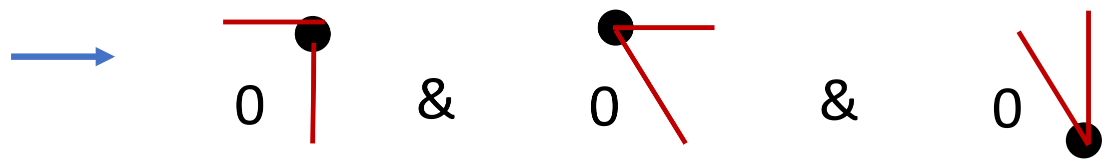

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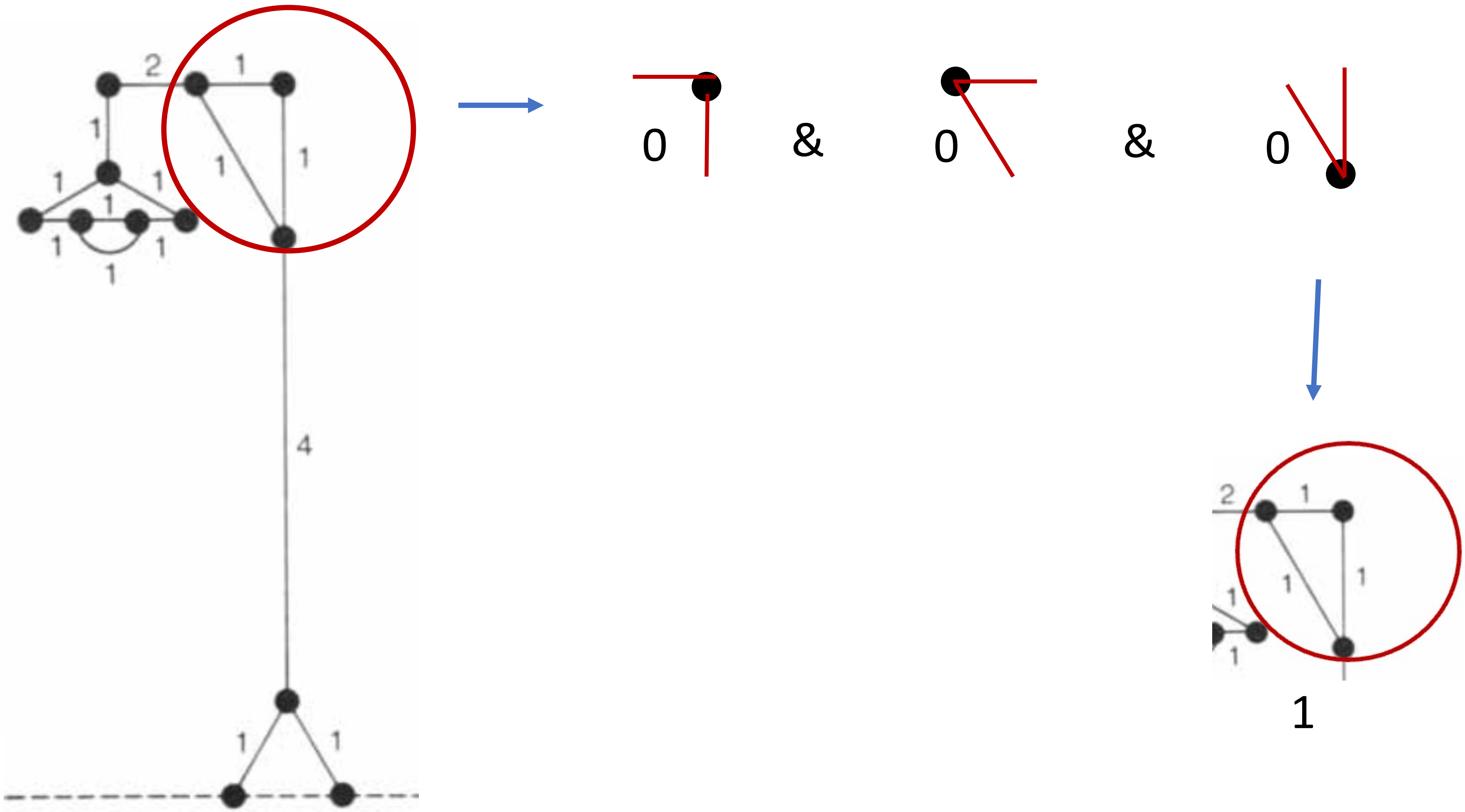
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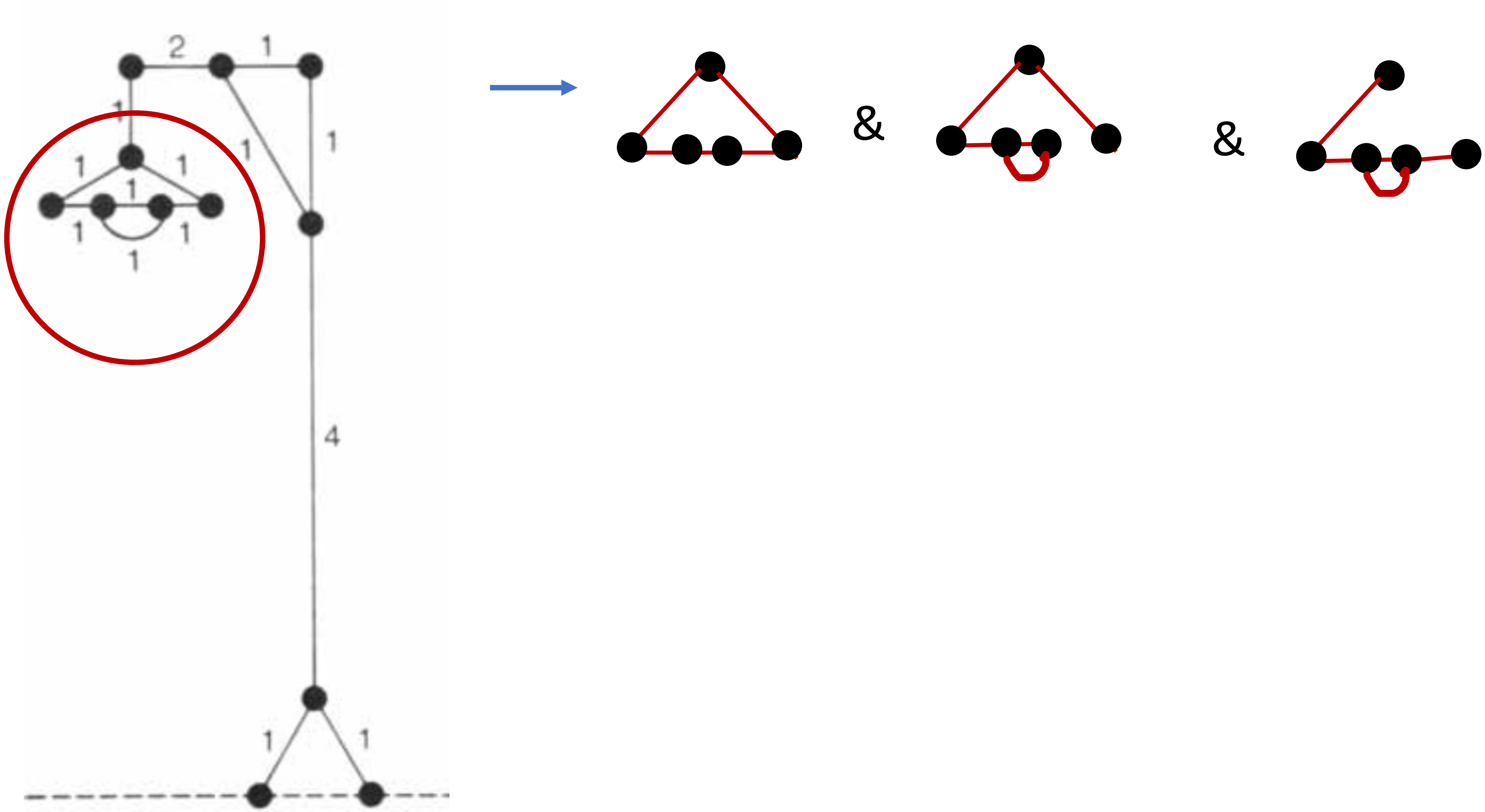

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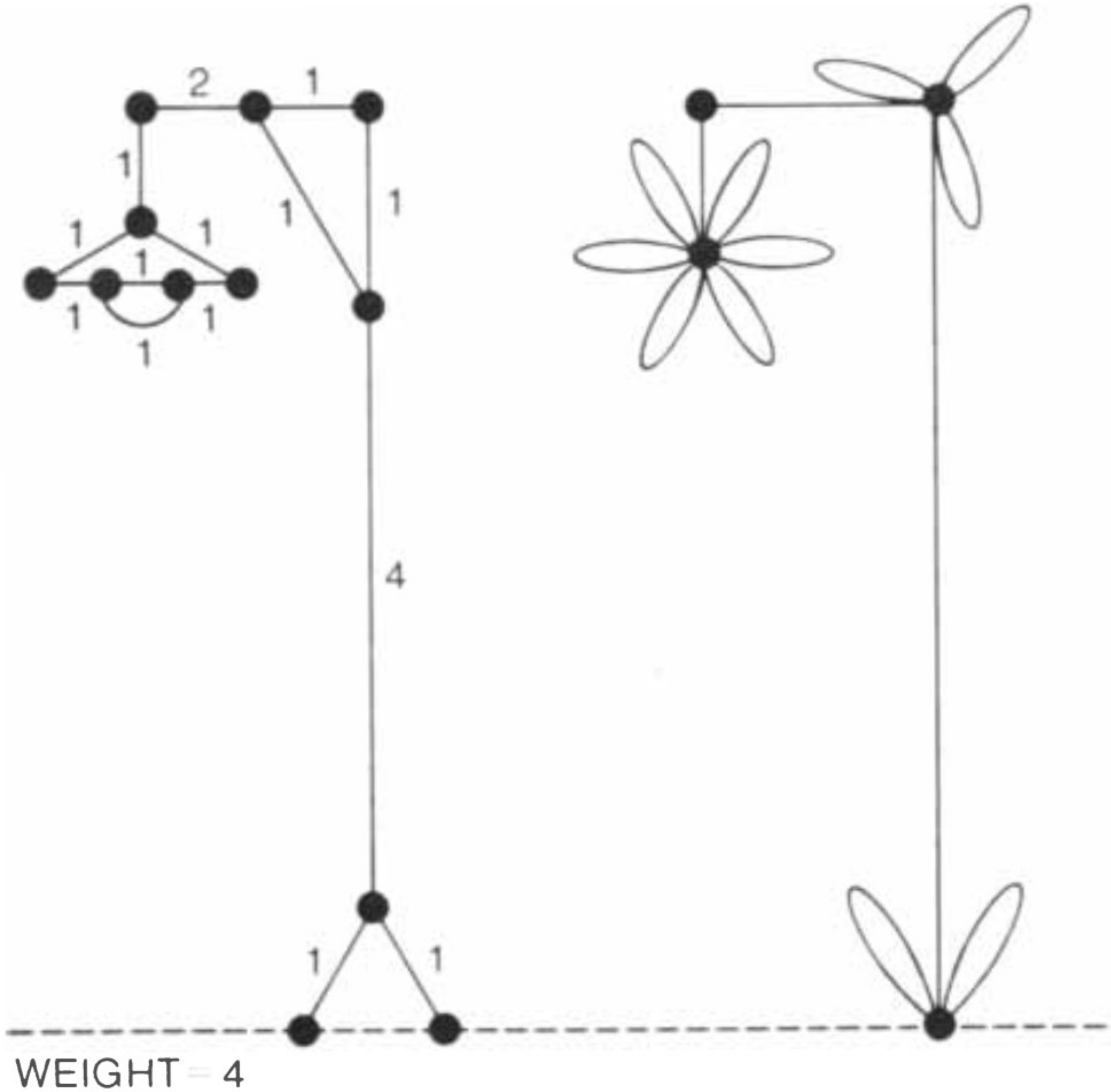
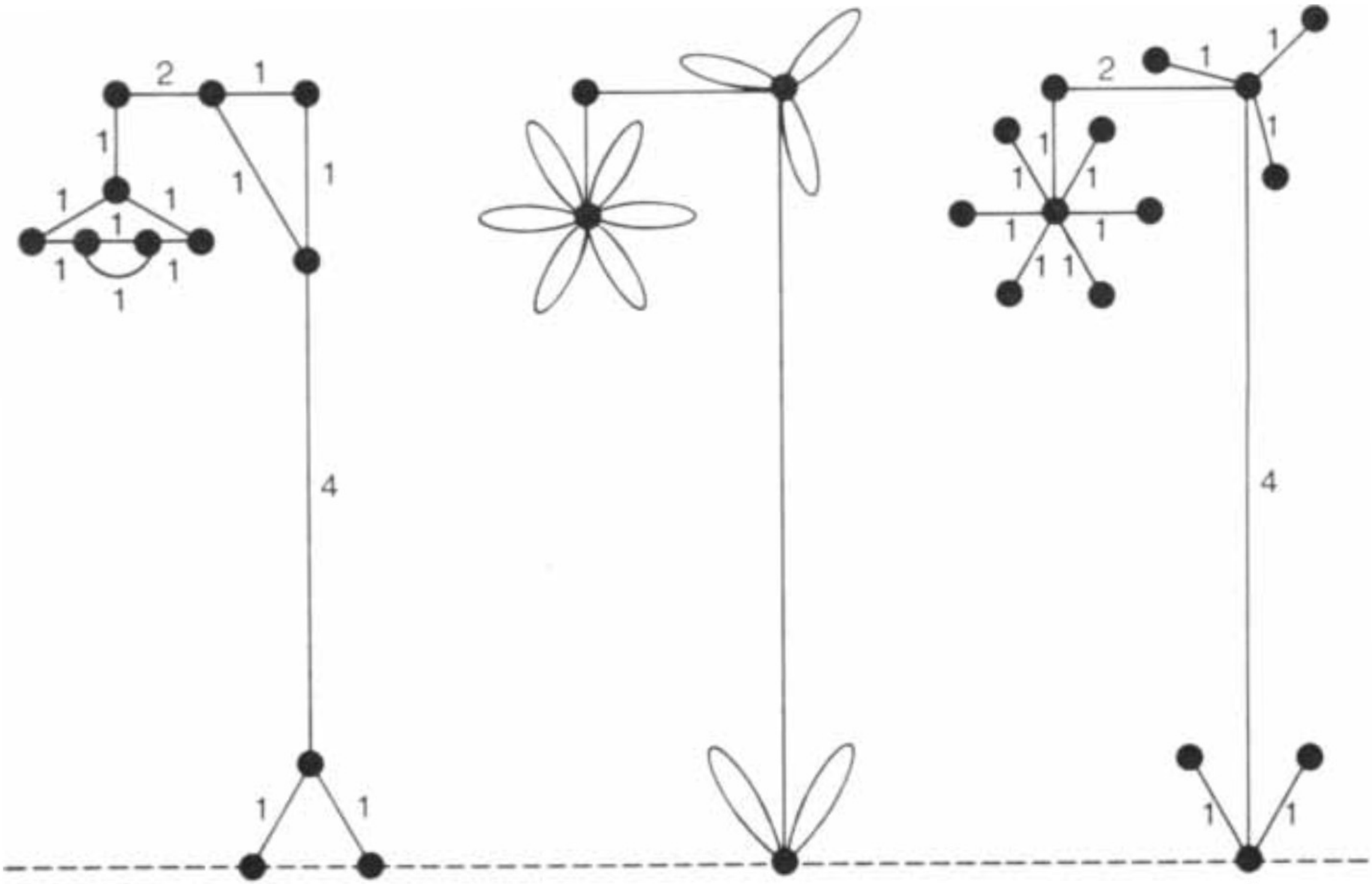
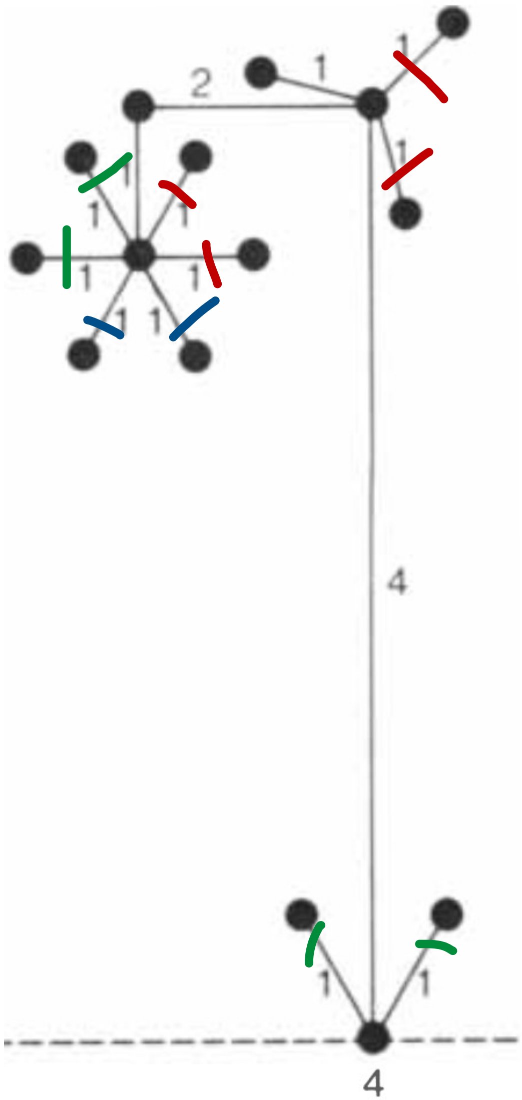


Merge nodes in cycle



WEIGHT = 4 4



2
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2
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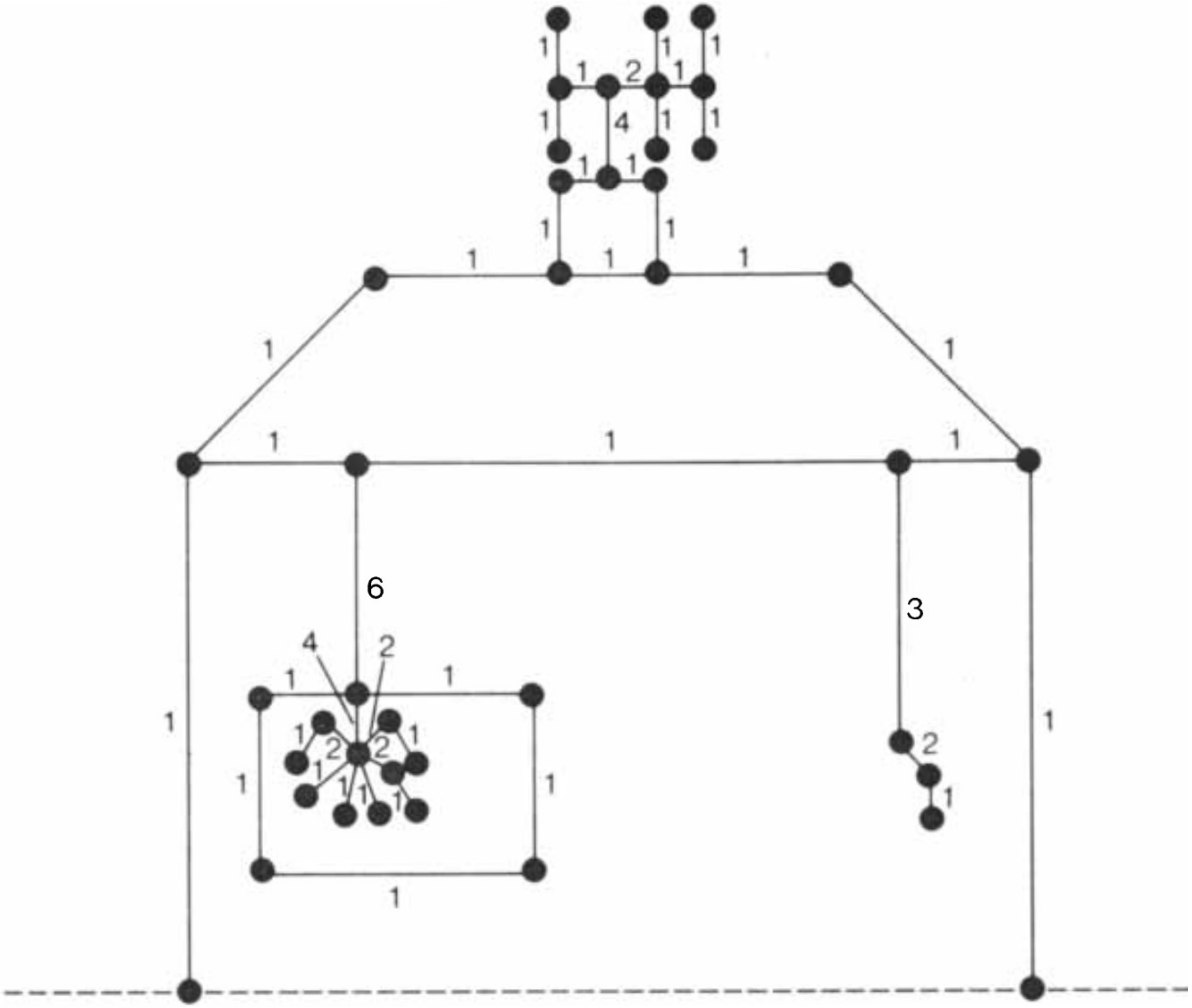
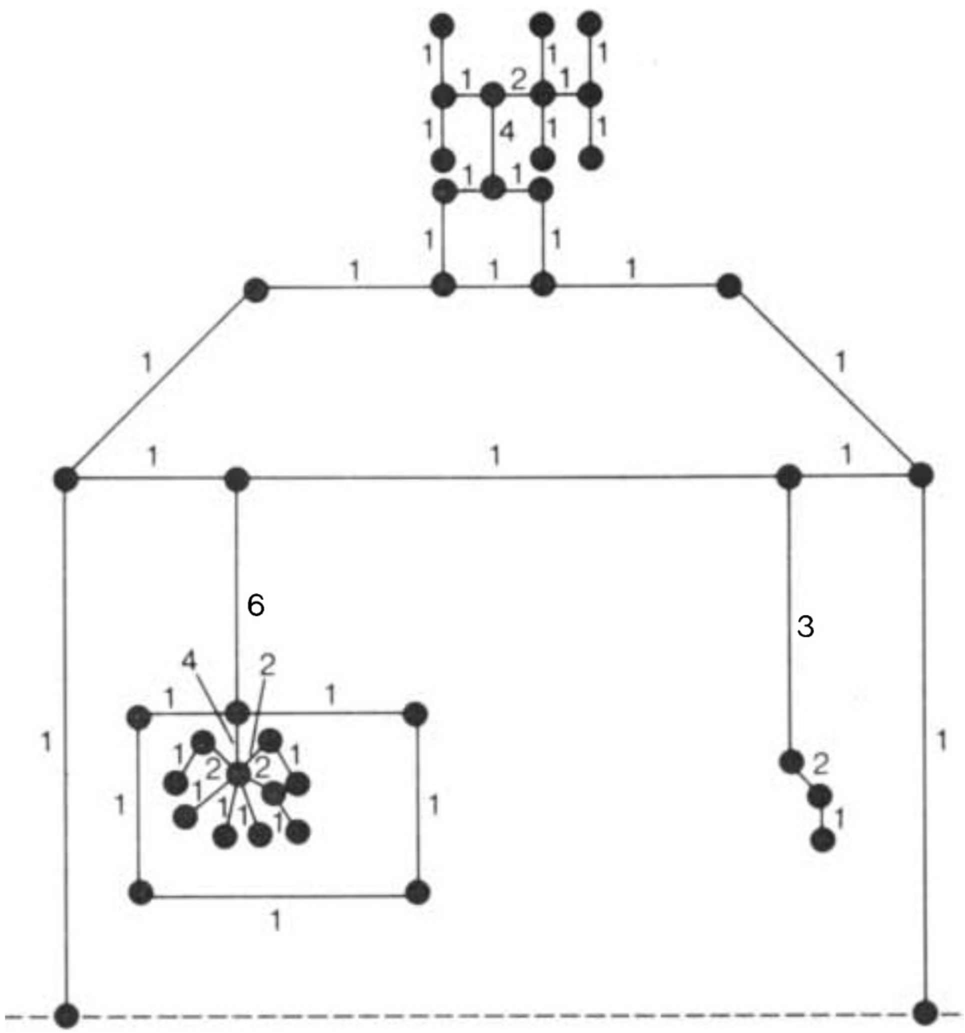
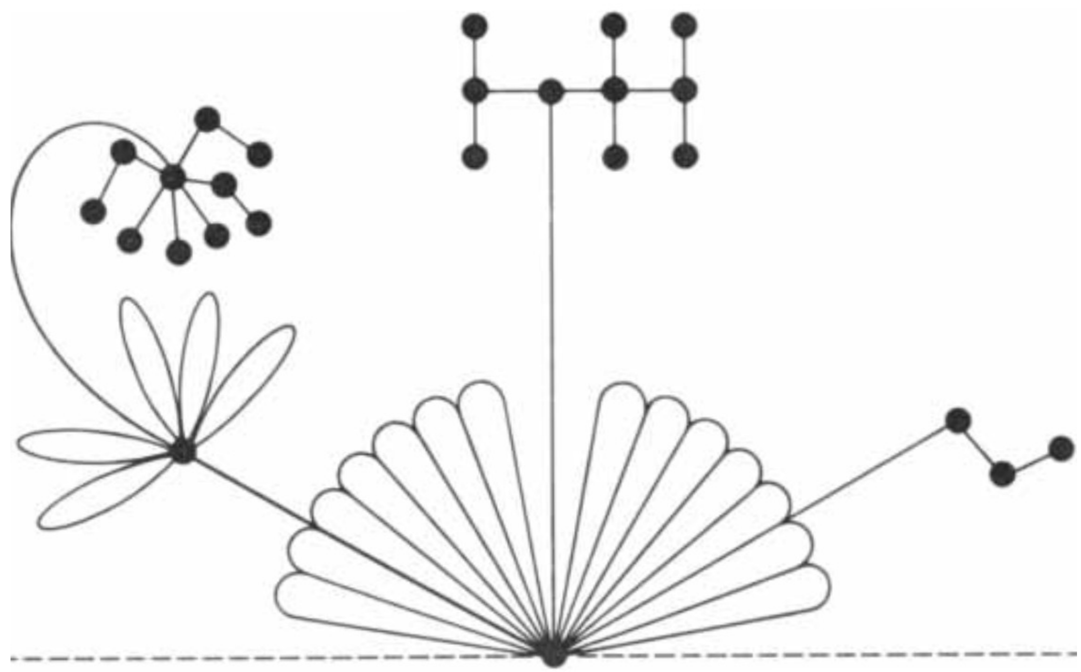
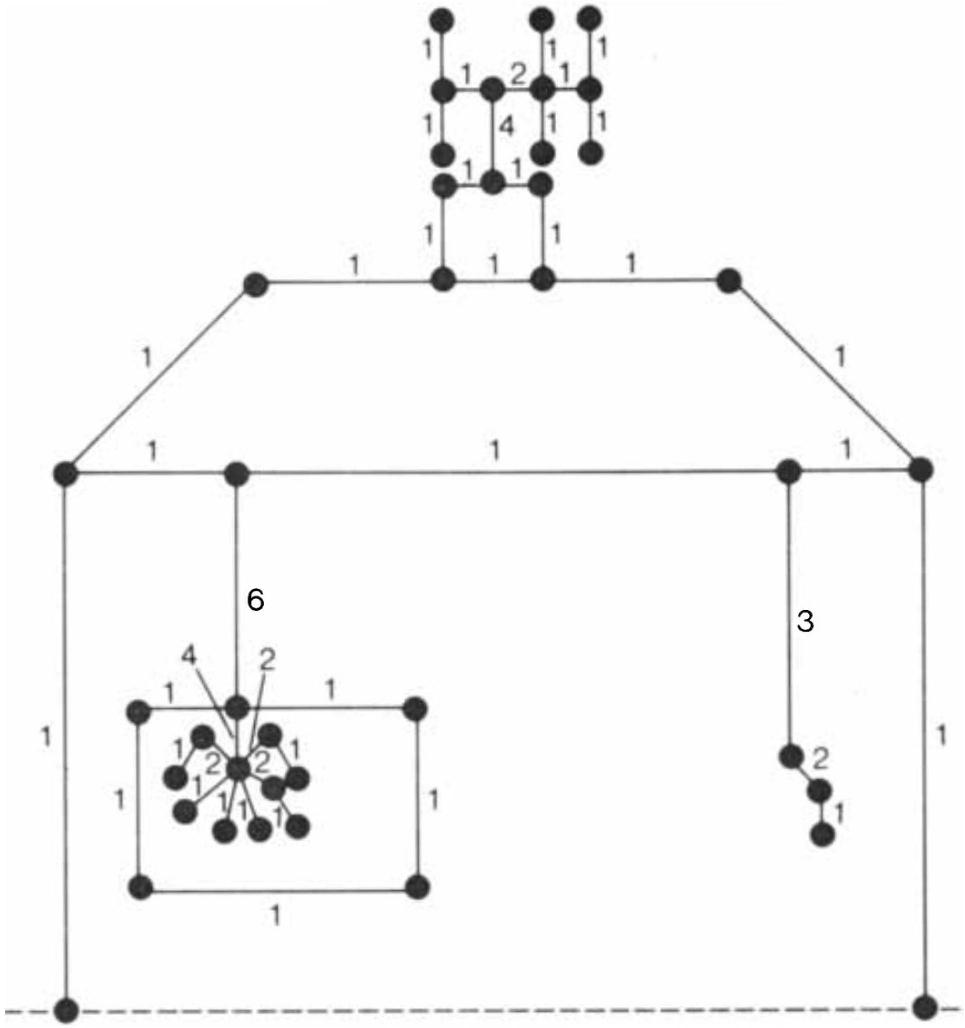
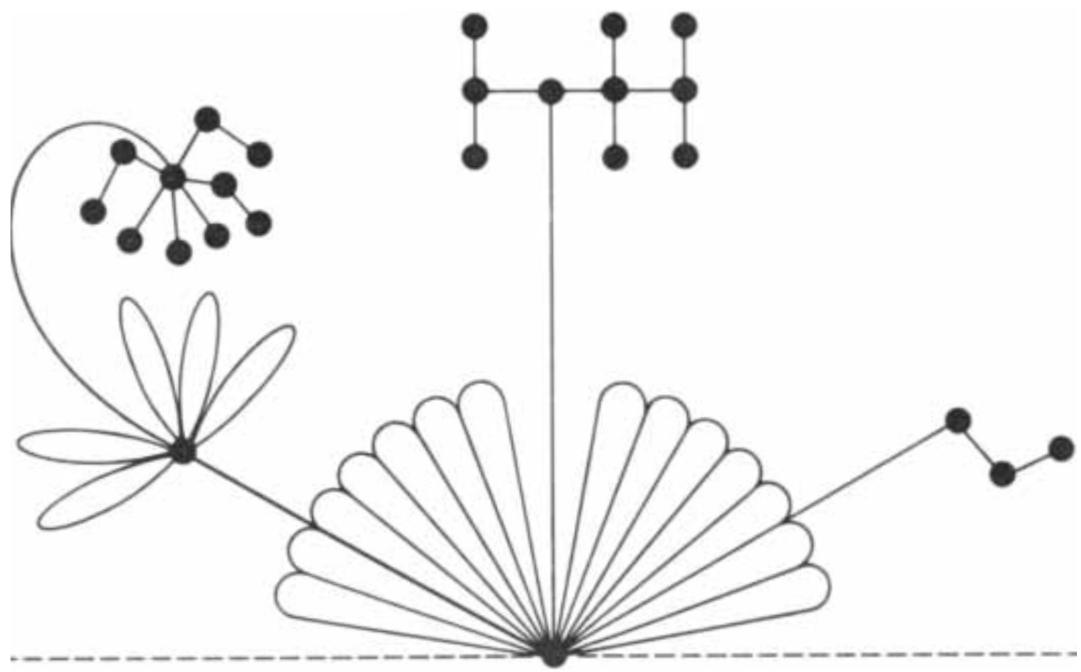
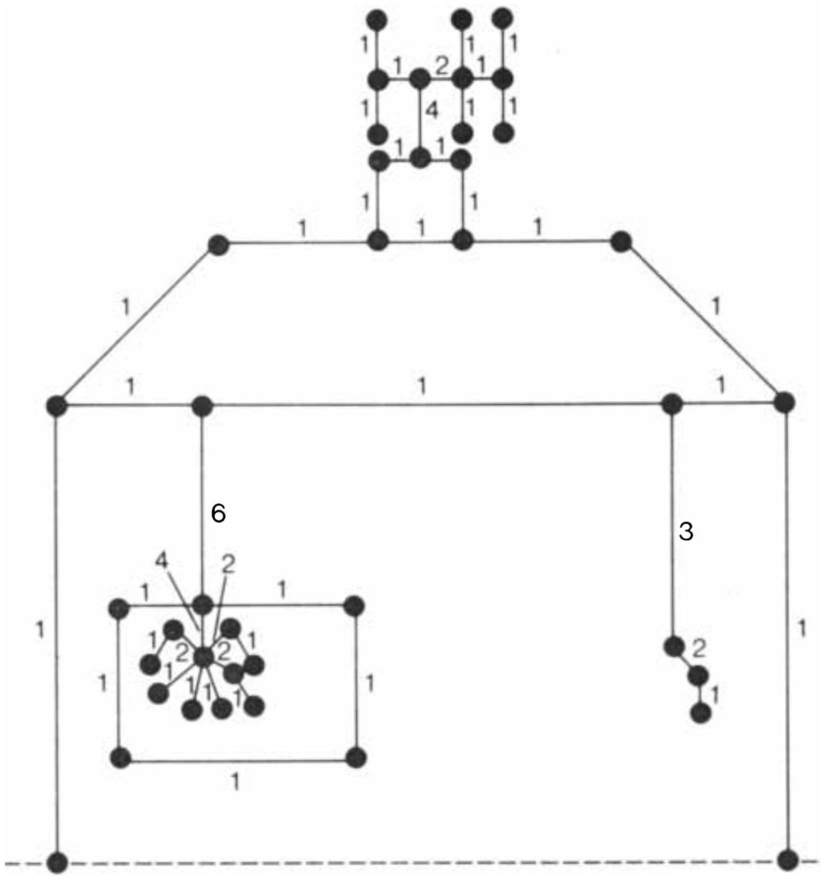


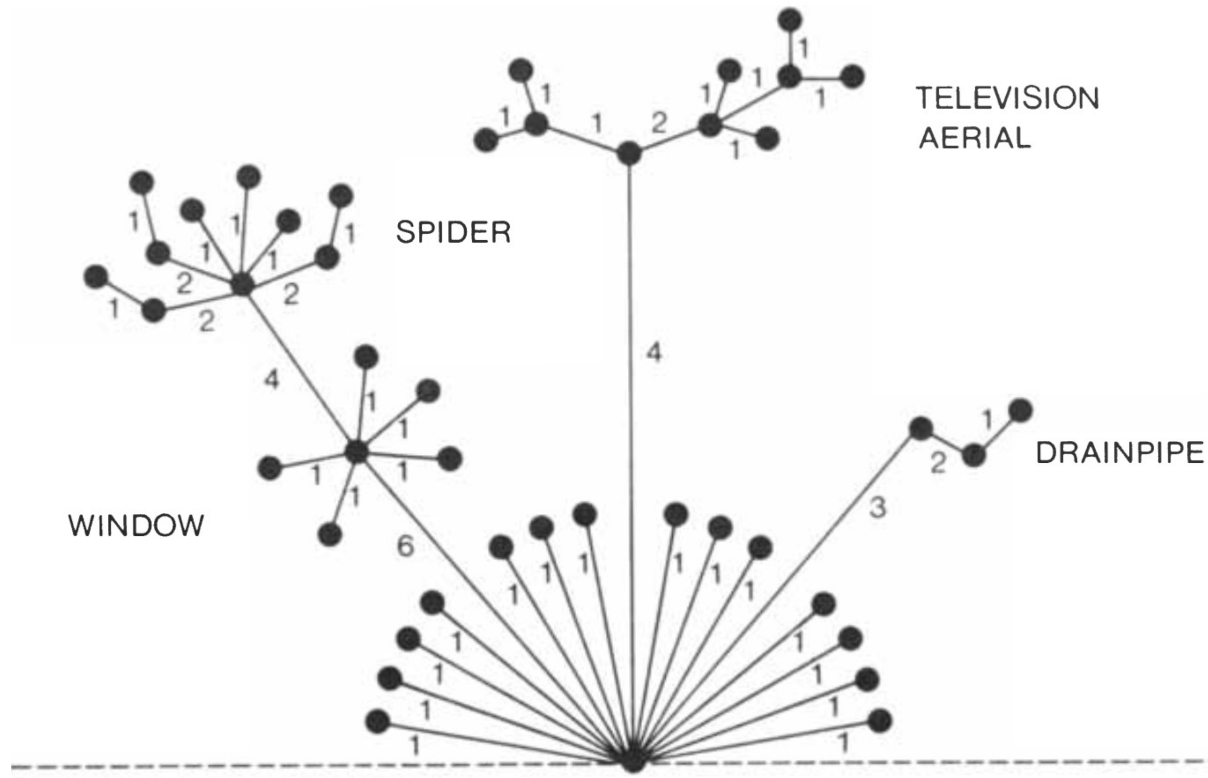




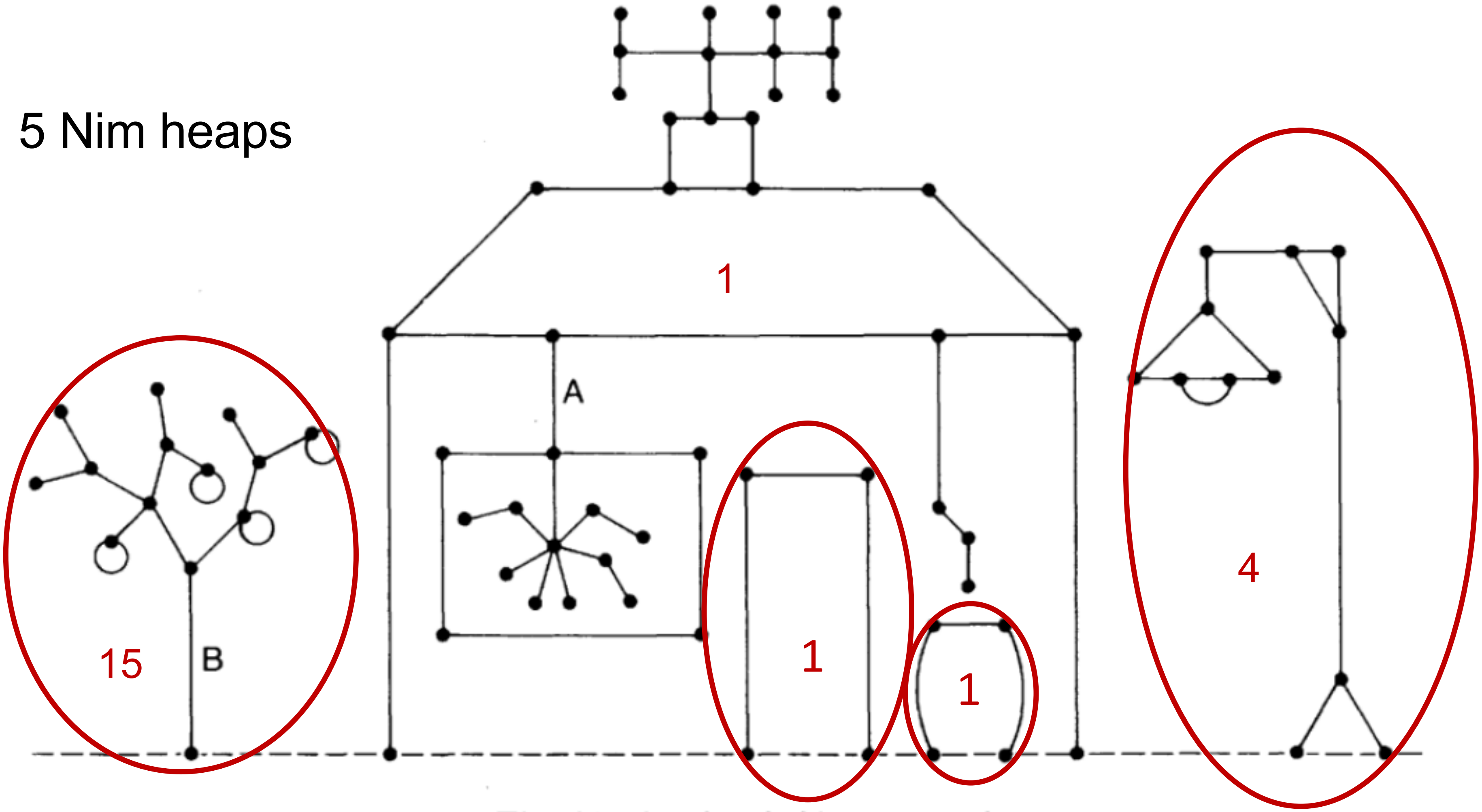






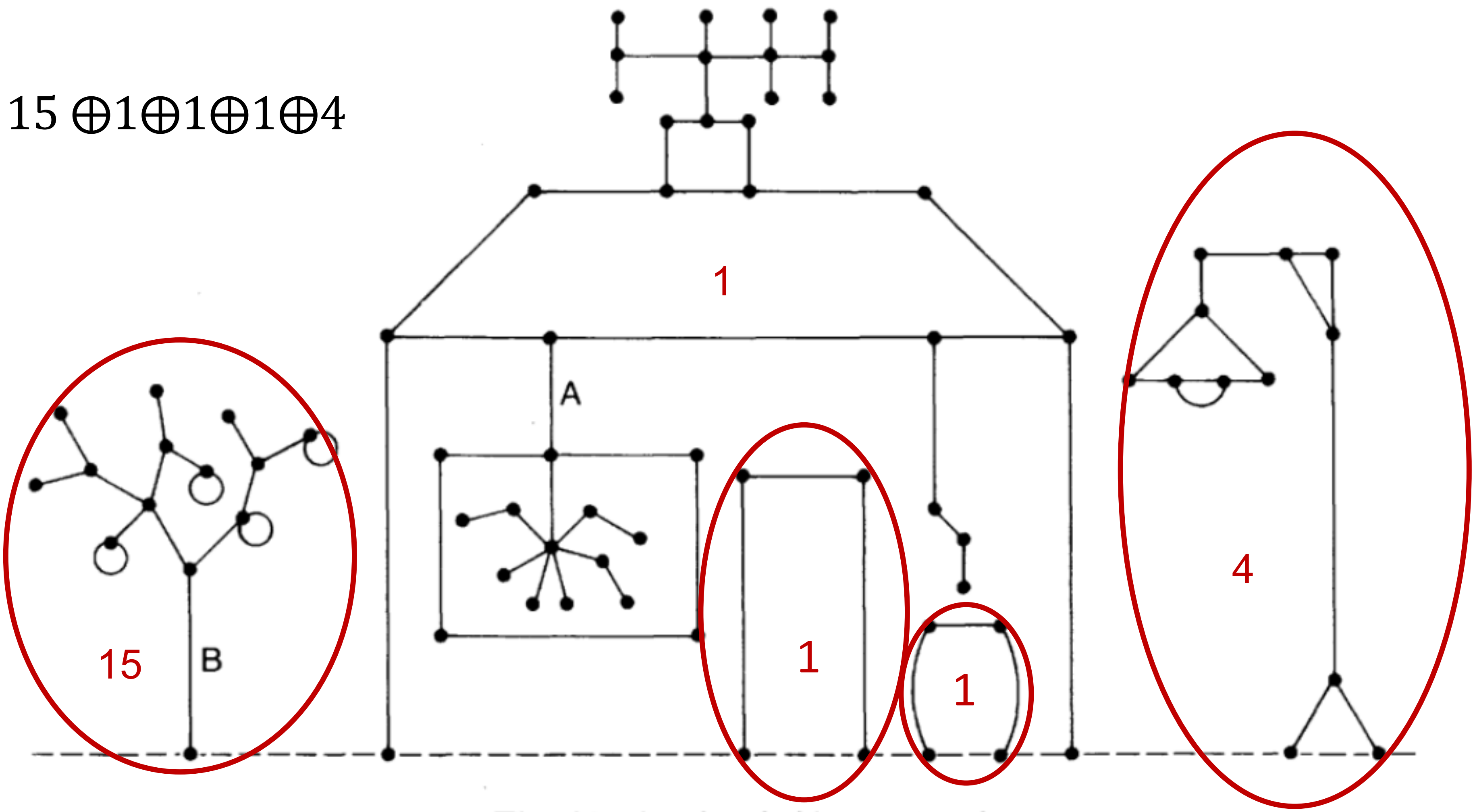

TELEVISION
AERIAL
SPIDER
DRAINPIPE
WINDOW
WEIGHT = 1

5 Nim heaps



The Hackenbush Homestead

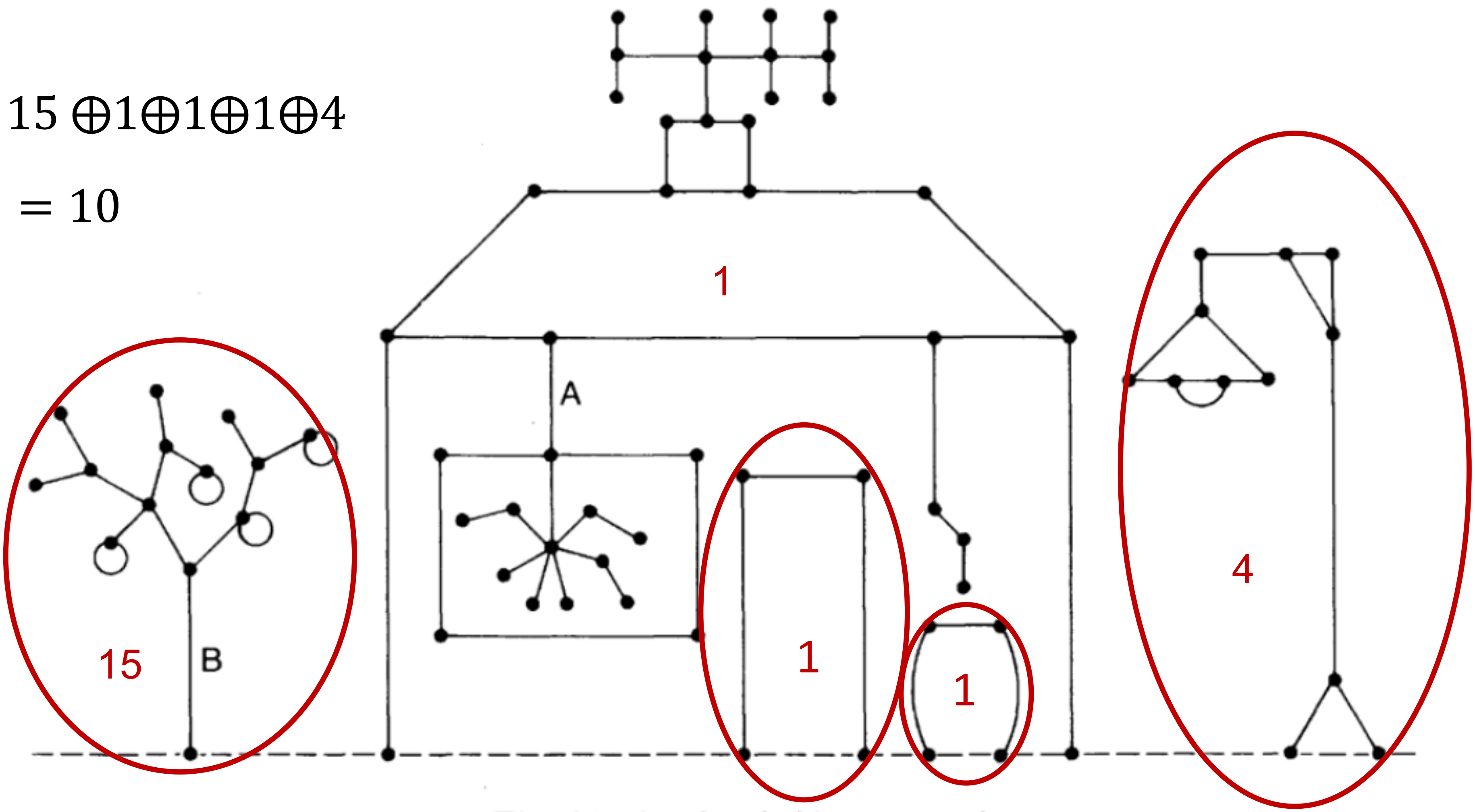
15 ⊕1⊕1⊕1⊕4



The Hackenbush Homestead

$15 \oplus 1 \oplus 1 \oplus 1 \oplus 4$

$= 10$



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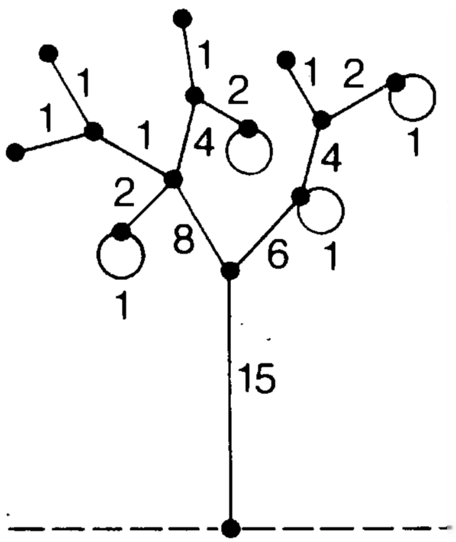
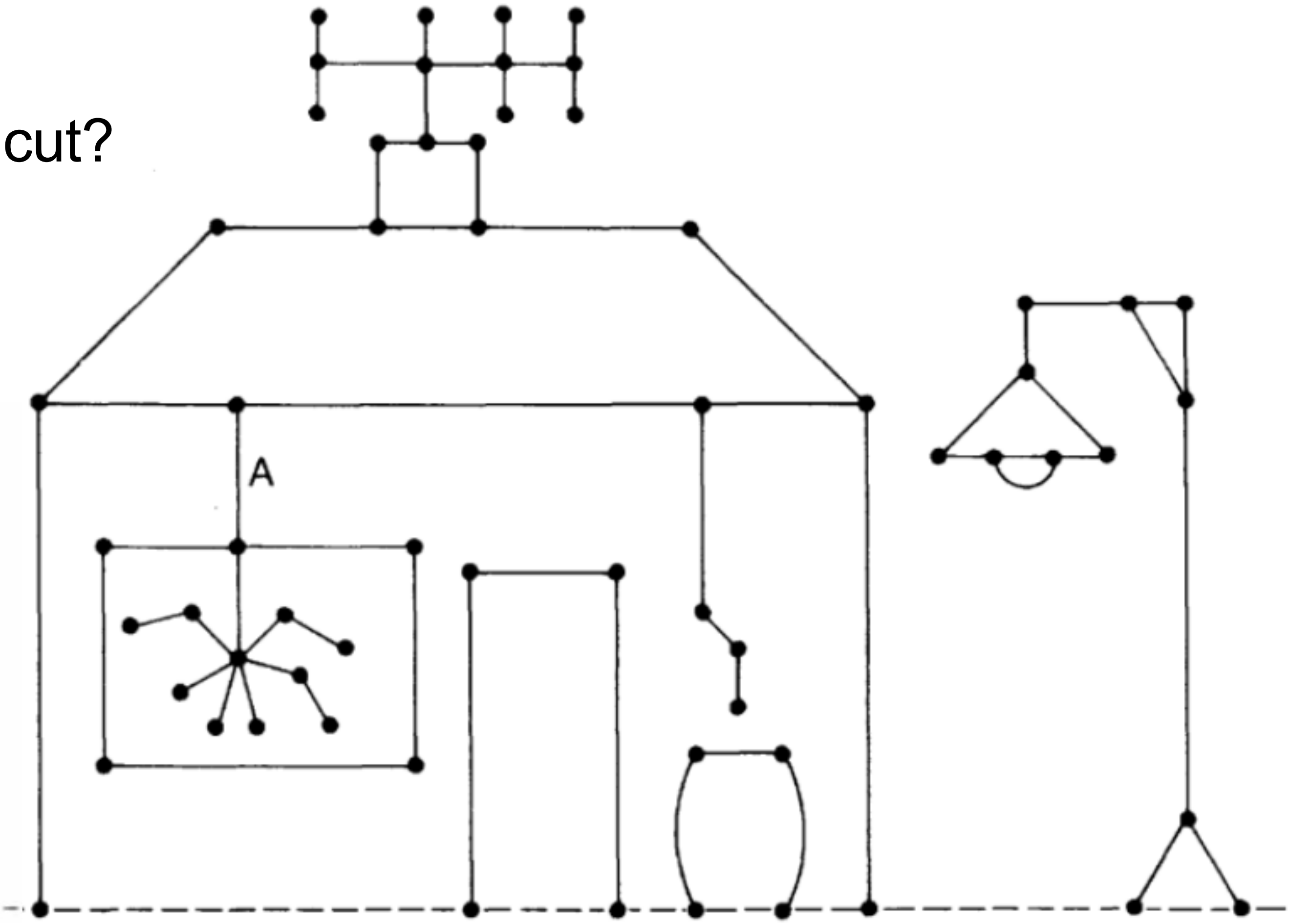
Where should you cut?



Branches are 8, 6



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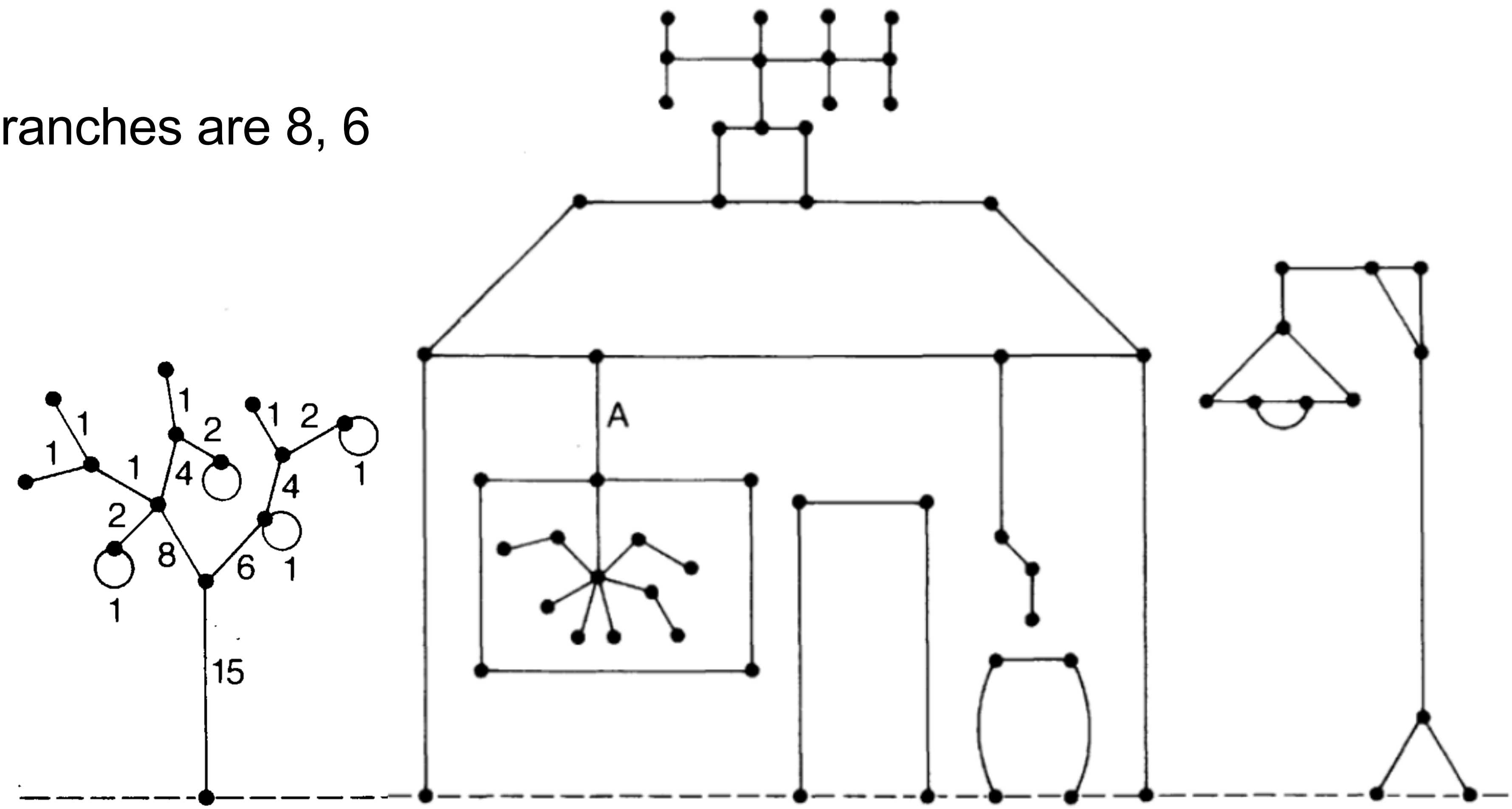
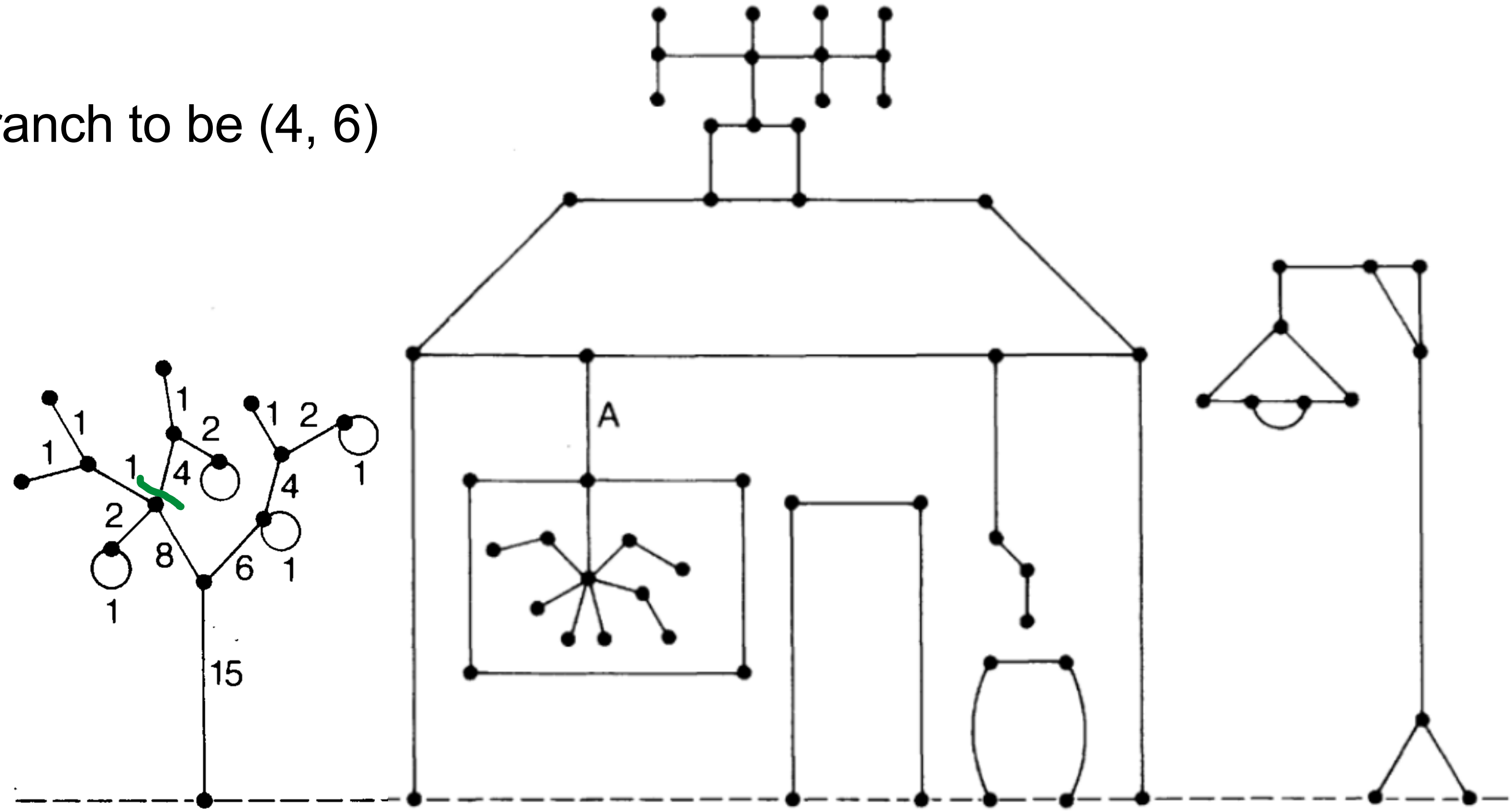
Branches are 8, 6



Want (4,6) or (8,2)

The Hackenbush Homestead

Branch to be (4, 6)



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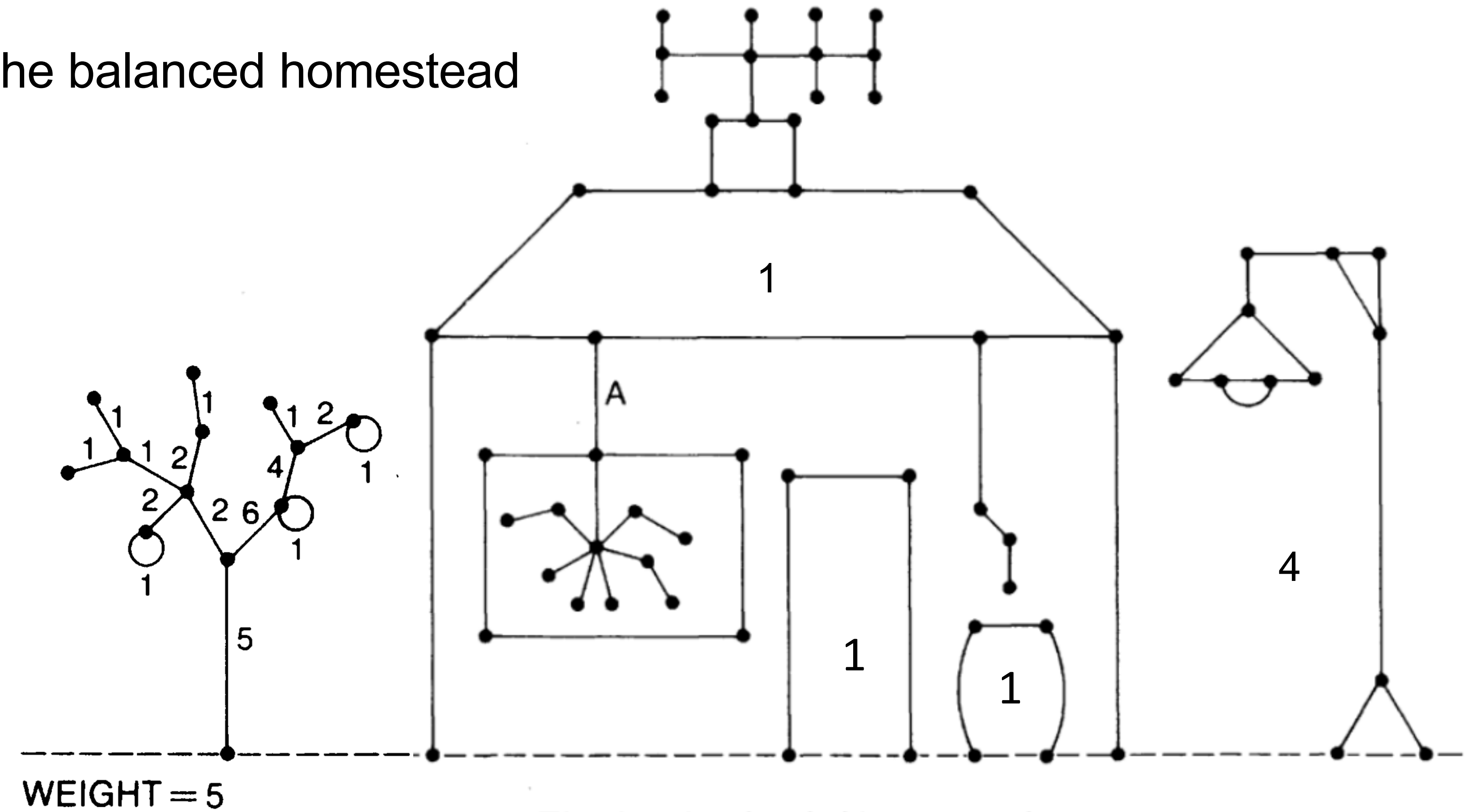
The balanced homestead



The Hackenbush Homestead