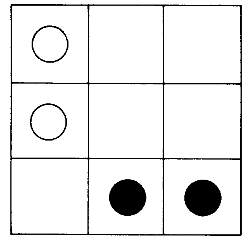
CHAPTER

TWELVE



Dodgem and Other Simple Games

Consider a two-person game with the following characteristics: (1) It is a game of perfect information; that is, both players have complete knowledge of the game's structure after every move. (2) The players move alternately. (3) Decisions are not made by chance. (4) The game ends after a finite number of moves with a win by one player. (No draw is possible.)

It is not hard to see that there must be a winning strategy for either the first player or the second. If the first player (henceforth called *A*) does not have a winning strategy, he must lose. This means that the second player, *B,* has a winning strategy. Does the argument apply if we rescind the requirement that the game end in a finite number of moves?

Curiously, it all depends on whether or not one accepts the "axiom of choice." This notorious axiom says that from any collection (finite or infinite) of nonempty sets, with no elements in common, you can form a new set by taking one element from each set. In the 1930s, Stefan Banach, Stanislaw Mazur, and Stanislaw Ulam discovered a type of infinite game in which neither *A* nor *B* has a winning strategy if the axiom of choice is accepted. Someone argued that this proves the Unitarian dogma that there is "at the most" one God, because if two gods could play such a game, neither could know a winning strategy and therefore neither could be called omniscient!

That, however, is by the way. Here we shall examine some new two-person nonchance games for which the rules are extremely simple and for which a winning strategy is either known or capable of being known. All but one of the games are played with counters on boards that are easily drawn on cardboard. Two differently colored sets of counters, such as go stones or small poker chips, will be useful for any reader who wants to play or to analyze the games.

An example of an almost trivial game of the nim type, but one with a strategy that is not immediately apparent, is played on the star pattern shown in Figure 68. Put a counter on each of the star's nine points. *A* and *B* take turns removing either one counter or any two counters joined by a straight line segment. The player who takes the last counter wins.

B can always win at star nim by a strategy based on the board's symmetry. Imagine that the black lines are strings. The pattern can be opened up to a circle that is topologically equivalent to the star. If *A* takes one counter from this circle, *B* takes the two counters that are directly opposite. If *A* takes two counters, *B* takes one counter that is directly opposite. In each case two sets of three counters are left. Now, whatever *A* takes from one set, *B* takes the corresponding counter or counters from the other set. Obviously *B* will get the last counter. If the reader plays a few games on the circle, translating each move to its equivalent on the star, he or she will soon see how to use the star's symmetry for playing the strategy.

In the late 1960s, G. W. Lewthwaite, of Thurso, Scotland, invented a delightful game with an artfully concealed "pairing strategy" that gives the second player a sure win. On a 5-by-5 square matrix place thirteen black

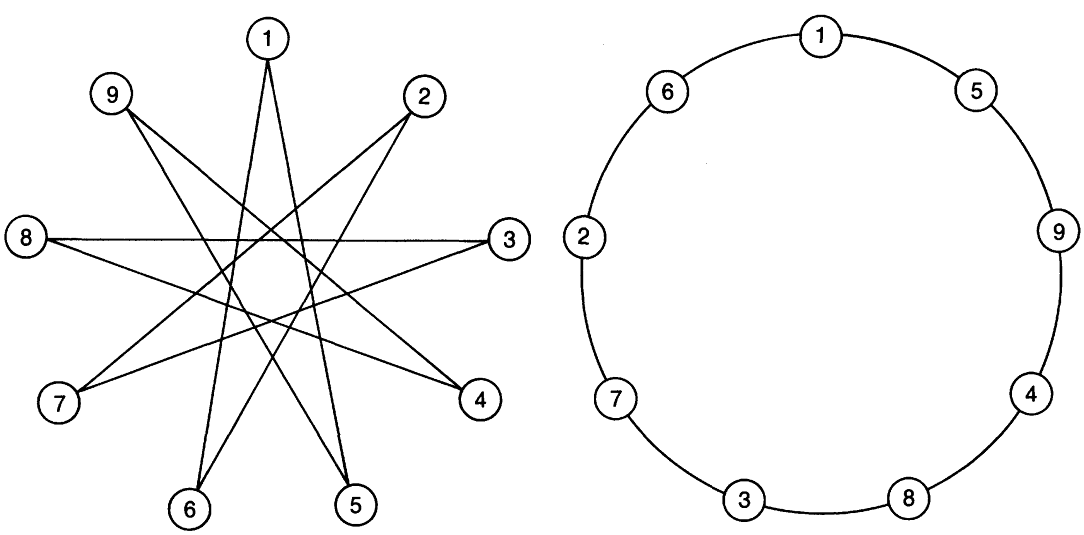


Figure 68 Star nim (left) and its winning strategy (right)

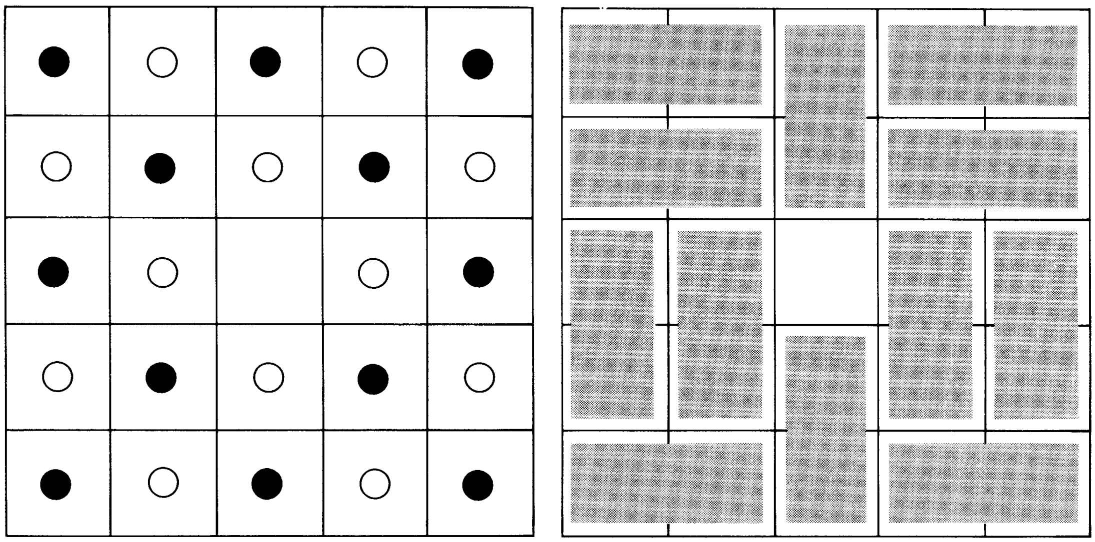


Figure 69 G. W. Lewthwaite's counter game (left) and a pairing strategy for Lewthwaite's game (right)

counters and twelve white counters in alternating checkerboard fashion. Any one of the black counters, say the one in the center, is removed (*see* Figure 69, left).

Player A controls the white counters and B the black. They take turns moving one of their counters orthogonally to the vacant square until a player loses by being unable to move. If the board is colored like a checkerboard, it is obvious that on each move, a counter goes to a square of different color and that no counter can be moved twice. The game, therefore, cannot go beyond twelve moves for each player. It may end before then, however, in a win for either player unless B plays rationally.

B's strategy is to imagine that the matrix, except for the initially vacant cell, is covered with twelve nonoverlapping dominoes. It does not matter how they are placed. Figure 69 (right) shows a sample pattern. Whenever A moves, B simply moves his counter that is on the domino A has just vacated. Since this ensures that B always has a move to follow a move by A, B is sure to win in twelve or fewer moves.

The game can be played not only with counters but also with square tiles or cubes that slide within a matrix surrounded by a rim. Suppose the rules are amended to allow either player at any time to move any number of adjacent counters (1 through 4) in a row or column provided that the two end counters are of his or her color. This is a splendid example of how an apparently trivial alteration of a rule can enormously complicate a game's analysis. Lewthwaite was unable to find a winning strategy for either player in this variant of his game.

Games based on the sliding of unit squares within a square matrix offer a plethora of unexplored possibilities. Lewthwaite proposes an attractive game that he calls meander. It uses twenty-four identical tiles placed in a 5-by-5 tray to form the pattern shown in Figure 70 (left). The players take turns sliding a single counter or a straight row or column of two, three, or four counters. The play continues until a player wins by creating a pattern in which at least three tiles form a continuous line or path that joins two edges (opposite or adjacent) of the tray. Figure 70 (right) shows a winning pattern, with the winning line indicated by the two arrows. The game is probably too complex for solving without a computer program, and perhaps too complex for solving even with one.

In 1972, when Colin Vout was a mathematics student at the University of Cambridge, he invented an intriguing counter game that he calls dodgem because it is so often necessary for a piece to dodge around enemy pieces. It is playable on a checkerboard of any size. Even the game on a 3-by-3 board is complicated enough to be interesting.

Two black counters and two white ones are initially placed as shown in Figure 71 (top). Black sits on the south side of the board and White sits on the west. The players alternately move a counter one cell forward or to their left or right, unless it is blocked by another counter of either color or by an edge of

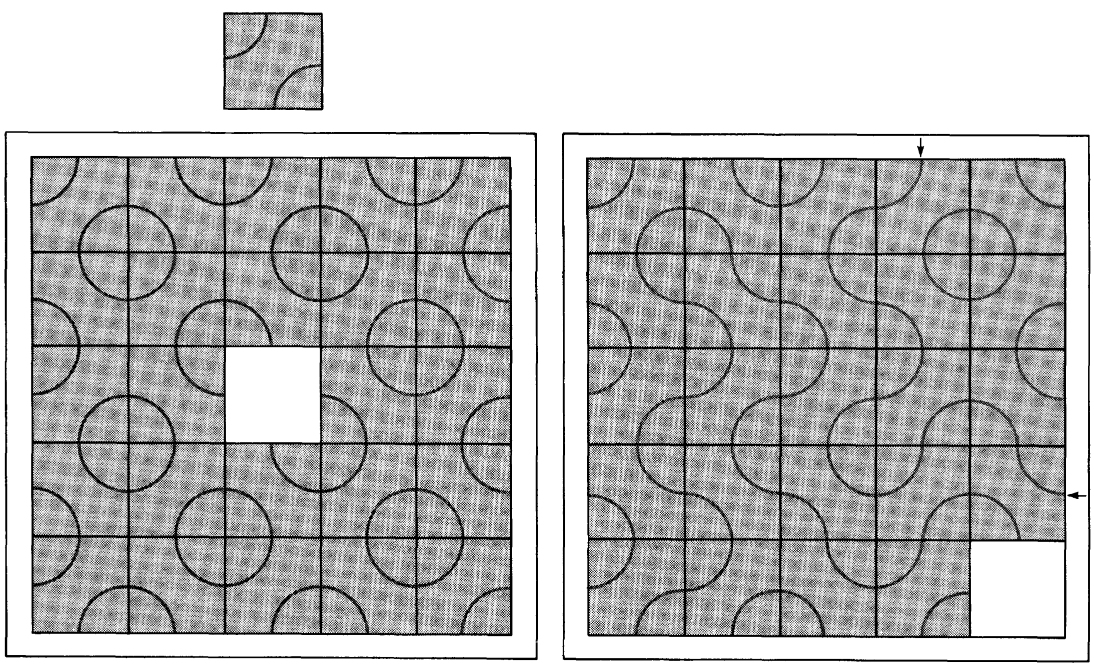


Figure 70 Meander, with example of pattern on a tile at top (left) and a possible winning pattern in meander (right)

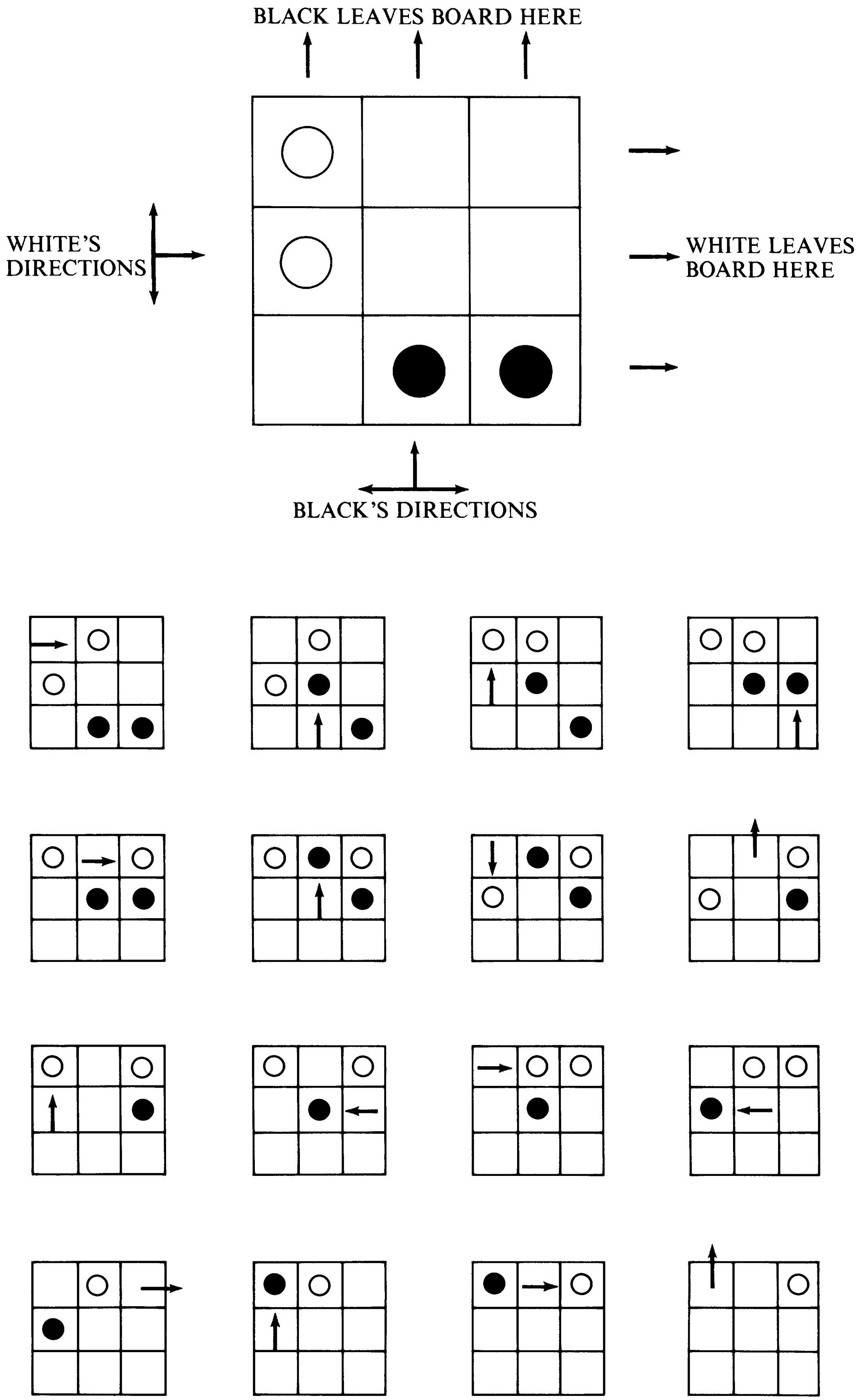


Figure 71 Colin Vout's dodgem (top) and a dodgem game won by Black (bottom)

the board. Each player's goal is to move all his pieces off the far side of the board. In other words, Black moves orthogonally north, west, or east and attempts to move both of his pieces off the north side of the board. White moves east, north or south and tries to move his pieces off the east side of the board.

There are no captures. A player must always leave his opponent a legal move or else forfeit the game. The first to get all his pieces off the board wins. The bottom of Figure 71 shows a typical game won by Black.

Vout assures me that the first player has the win on the order-3 board, but as far as I know, no games on higher-order boards have yet been solved. On a board of side n, each player has $n - 1$ pieces placed on the west and south edges, with the southwest corner cell vacant. Played with seven checkers or pawns of one color and seven of another color on the standard order-8 checkerboard or chessboard, it is a most enjoyable game.

Piet Hein's now classic game of hex (see Chapter 8 of my *Scientific American Book of Mathematical Puzzles & Diversions*, Simon & Schuster, 1959) remains unsolved, except for small boards. For readers unfamiliar with the game, it is played on an n-by-n rhombus of hexagons such as the order-4 board shown in Figure 72. White opens by placing a white counter on a cell. Black follows with a black counter. They take turns placing counters on vacant cells (there are no moves or captures) until a player wins by forming a chain of adjacent counters that joins his side of the board to the opposite side, White by joining the north and south edges, Black by joining the east and west edges.

It is easy to see that no draw is possible. There is a famous proof by John F. Nash (who independently invented hex) that on a rhombus of any size, the first player has a winning strategy, although the proof gives no hint of what the strategy is.

Suppose White allows Black to tell him where he must make his first move. Can White still always win if he plays rationally? This modified version of hex has been called Beck's hex after Anatole Beck, who both proposed and solved it. Writing on hex in Chapter 5 of *Excursions into Mathematics,* Beck shows that Black can always win if he tells White to open by taking an acute corner cell. In other words, such an opening is a sure loss for White, although Beck's proof does not provide Black's winning strategy. However, as a footnote comments, it "wrecks Beck's hex."

What about *misère,* or reverse, hex, known as rex, in which the first player to join his sides loses? As is so often the case in two-person games, the reverse game proves to be much harder to crack. No general strategy is known, although Robert O. Winder, in unpublished arguments, has shown the existence of a first-player winning strategy in rex of even order and a second-player winning strategy for all odd orders. More recently Ronald J. Evans has

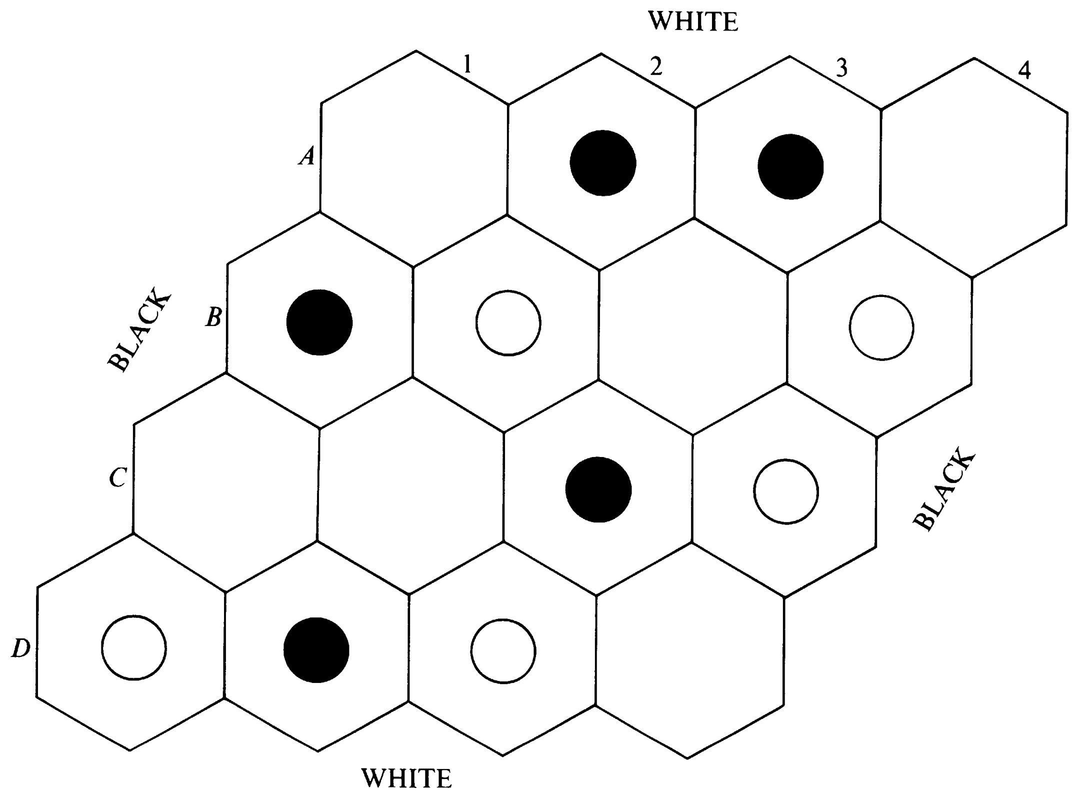


Figure 72 Rex, a reverse hex game, with White to play and win

carried Winder's arguments a step further by showing that on even-order boards there is a winning strategy if White opens in the acute corner.

Rex on the order-2 rhombus is trivial, and it is not difficult to analyze exhaustively on the order-3 rhombus. Play on the order-4 rhombus is so complicated, however, that even though it is known that an acute-corner opening initiates a win, the strategy itself remains unformulated. The position shown in Figure 72 is an order-4 rex problem composed by Evans. Can the reader determine White's only correct move?

Here is an even simpler game for which no general strategy is known. It is played on a 1-by-n board (a single row of n squares) with counters that are all alike. A and B take turns placing a counter until one player wins by getting three counters adjacent. Could anything be simpler? A can always win when n is odd by first taking the center cell, then playing symmetrically opposite the opponent thereafter. For even n, however, things are not so simple. On most even rows, A seems to have the win, but not necessarily, and the exceptions follow no known rule. Take $n = 6$, for example. The reader may enjoy working it out to see who has the win.

John Horton Conway has pointed out that this game is equivalent to a game I called 1-by-n cram in a column reprinted as Chapter 19 of my *Knotted*

Doughnuts and Other Mathematical Entertainments, (W. H. Freeman and Company, 1986) except that it is played with trominoes instead of dominoes. It is easy to see the isomorphism. In playing the game as described above, it is obviously disastrous to place a counter either next to another counter or one cell from it, since either move gives the opponent an instant win. Hence we might as well prohibit both moves. An easy way to do it is to require that each play consist of a triplet of adjacent counters, which is the same as placing a tromino on the field. (The middle of the triplet corresponds to placing a single counter, and the ends of the triplet enforce the two new rules.) The winner is the player who places a tromino last. (To complete the equivalence, we must allow the placing of a tromino at either end of the field so that it extends one cell beyond the end.) Of course, the game can also be played by forming a row of *n* counters and by removing them by alternate moves of taking three adjacent counters.

This triplet version of cram is considered in a classic paper by Richard K. Guy and Cedric A. B. Smith, "The G-Values of Various Games." Because it is coded as game .007, it has been called the James Bond game. Elwyn Berlekamp has computer-analyzed the game to very high even *n* without finding any periodicity in the Grundy numbers, which means that no one is even close yet to a general rule. The *misère* version of 1-by-*n* tromino cram, regardless of the parity of *n,* is also unsolved.

Ulam has proposed extending the counter form of tromino cram to a square matrix. The players take turns placing single counters until one player wins by getting three in a row orthogonally or diagonally. As before, odd-order fields are trivial because the first player wins by taking the center, then playing symmetrically until the opponent offers a win. On even-order boards the order 4 is trivial, but no one yet knows who has the win on orders 6 or 8. Figure 73, supplied by Ulam in a letter, shows a position on the order-6 board for which the next player must lose.

Here again we can play an equivalent game by alternately placing polyominoes, in this case squares of nine cells, but it is not very convenient because in addition to allowing the pieces to extend into a unit border around the field, we must also allow them to overlap one another by just two cells (corner and side). No one has even begun to find a general strategy for the game in standard or reverse form.

ANSWERS

The answers to the chapter's two problems are that the second player has the win on 1-by-6 tromino cram, and White wins the 4-by-4 rex (reverse hex)

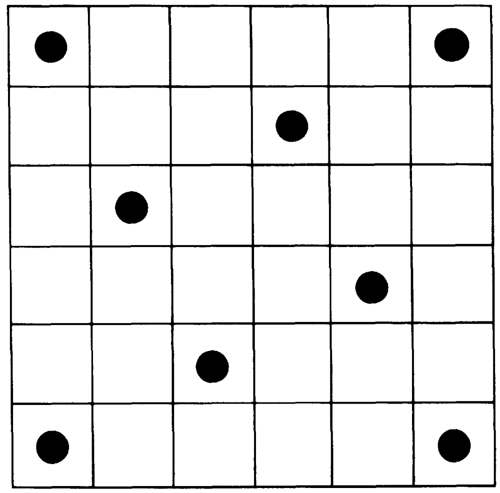


Figure 73 Stanislaw Ulam's triplet game

game by taking the cell at the intersection of row *C* and diagonal 1. Although no winning strategy for the first player of order-4 rex is known, a winning second-player pairing strategy for the order-5 game has been found by David L. Silverman. All higher orders remain unsolved.

G. W. Lewthwaite's game generalizes in an obvious way to rectangular boards of any size and shape. If the rectangle has an odd number of cells, the second player wins; if it has an even number of cells, the first player wins. (In the latter case the domino covering strategy includes the initially vacant cell.)

Karl Fulves has suggested that instead of visualizing a domino pattern, you play the game with counters secretly marked so that you can place them on the board in orientations that group them in pairs. For example, a small pinhole on the rim of a pawn or a checker would enable you to orient the pieces so that the pinholes of each pair are adjacent. You could then play the domino strategy without having to remember a domino pattern. If coins are used, designate a spot on the rim of each side of each coin, say the N in the ONE on a penny, as your mark and orient the coins accordingly.

ADDENDUM

David Fremlin and Dennis Rebertus independently wrote computer programs that verified the first player's win in order-3 dodgem. White wins only by first moving his piece at the corner. A full analysis of the order-3 game is in *Winning Ways*. Order-4 dodgem is still unsolved.

	3	4	5	6	7	8	9	10	11	12
3	2,2	*	2,3	*	2,4	2,5	2,5	*	2,6	3,3
4		3,3	*	3,4	4,1	*	*	3,8		
5			3,3	4,1	3,4	3,5	3,5			
6				4,4						

Figure 74 John Beidler's results for Stanislaw Ulam's triplet game

John Beidler, who heads the computer science department at the University of Scranton, found by computer that Stanislaw Ulam's triplet game in standard play on a 6-by-6 field is a win for the first player only if his first move is on one of the four central cells. Beidler generalized the game to rectangular boards and obtained the results shown in Figure 74. The numbers give winning moves by row and column for the first player. The asterisks indicate a win for the second player. If the game is played in reverse form, Beidler found that the second player has the win on 3-by-3, 3-by-4, 3-by-5, and 4-by-4 boards. For tromino cram, played in reverse on a 1-by-n field, with n less than 19, Beidler proved first-player wins for n = 4, 5, 7, 8, 11, 14, 15, 17, and 18, and second-player wins for all other values.

B I B L I O G R A P H Y

"The G-Values of Various Games." R. K. Guy and C. A. B. Smith, in *Proceedings of the Cambridge Philosophical Society* 52, July 1956, pp. 514–526.

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"Reverse Cram with Block Sizes Not Exceeding 13." R. J. Evans, in *Delta* 6, 1976, pp. 57–76.

"Dodgem." *Winning Ways.* E. R. Berlekamp, J. H. Conway, and R. K. Guy, pp. 685–688. Academic Press, 1982.