CHAPTER EIGHTEEN

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# Bridg-it and Other Games

*Man has never shown more ingenuity than in his games.*
— Leibniz, in a letter to Pascal

MATHEMATICAL games such as ticktacktoe, checkers, chess and go are contests between two players that (1) must end after a finite number of moves, (2) have no random elements introduced by devices such as dice and cards, (3) are played in such a way that both players see all the moves. If a game is of this type and each player plays "rationally" — that is, according to his best strategy — then the outcome is predetermined. It will be either a draw or a certain win for the player who makes the first move or the player who makes the second move. In this chapter we shall first consider two simple games for which winning strategies are known, then a popular board game for which a winning strategy has just been discovered and a class of board games not yet analyzed.

Many simple games in which pieces are placed on or removed from a board lend themselves to what is called a symmetry strategy. A classic example is the game in which two players take turns placing a domino anywhere on a rectangular board. Each domino must be put down flat, within the border of the rectangle and without moving a previously placed piece. There are enough dominoes to cover the board completely when the pieces are packed side by side. The player who puts down the last domino wins. The game cannot end in a draw, so if both sides play rationally, who is sure to win? The answer is the player who puts down the first domino. His strategy is to place the first domino exactly at the center of the board [*see Fig. 110*] and thereafter to match his opponent's plays by playing symmetrically opposite as shown. It is obvious that whenever the second player finds an open spot, there will always be an open spot to pair with it.

The same strategy applies to any type of flat piece that retains the same shape when it is given a rotation of 180 degrees. For example, the strategy will work if the pieces are Greek crosses; it will not work if they have the shape, say, of the letter T. Will it work if cigars are used as pieces? Yes, but because of the difference in shape between the ends the first cigar must be balanced upright on its flat end! It is easy to invent new games of this sort, in which pieces of different shapes are alternately placed on variously patterned boards according to prescribed rules. In some cases a symmetry strategy provides a win for the first or second player; in other cases no such strategies are possible.

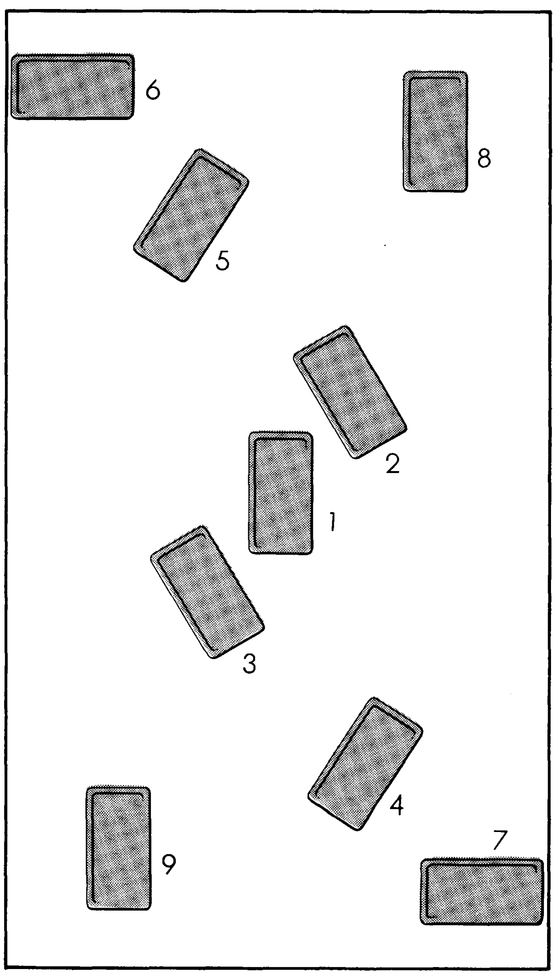


***FIG. 110***
***A domino board game.***

A different type of symmetry play wins the following game. Any number of coins are arranged in a circle on the table, each coin touching two of its neighbors. Players alternately remove either one coin or two touching coins. The player who takes the last coin wins. In this case it is the player who makes the second move who can always win. After the player who makes the first move has taken away one or two coins, the remaining coins form a curved chain with two ends. If this chain contains an odd number of coins, the player who makes the second move takes the center coin. If it contains an even number, he takes the two center coins. In both cases he leaves two separate chains of equal length. From this point on, whatever his opponent takes from one chain, he duplicates the move by taking one or two coins from the other chain.

Both this and the preceding strategy are examples of what game theorists sometimes call a pairing strategy: a strategy in which the plays are arranged (not necessarily in symmetrical fashion) in pairs. The optimal strategy consists of playing one member of the pair whenever the opponent plays the other member. A striking example of a pairing strategy is provided by the topological game of Bridg-it, placed on the market in 1960 and now a popular game with children. The reader may remember that Bridg-it was introduced in *Scientific American* in October 1958 as "the game of Gale"; it was devised by David Gale, a mathematician at Brown University.

A Bridg-it board is shown in Figure 111. If it is played on paper, one player uses a black pencil for drawing a straight line to connect any pair of adjacent black spots, horizontally or vertically but not diagonally. The other player uses a red pencil for similarly joining pairs of red spots. Players take turns drawing lines. No line can cross another. The winner is the first player to form a connected path joining the two opposite sides of the board that are his color. (The commercial Bridg-it board has raised spots and small colored plastic bridges that are placed between spots.) For many years a proof has been known that there is a winning strategy for the player who makes the first move, but not until early this year was an actual strategy discovered.

It was Oliver Gross, a games expert in the mathematics department of the Rand Corporation, who cracked the game. When I

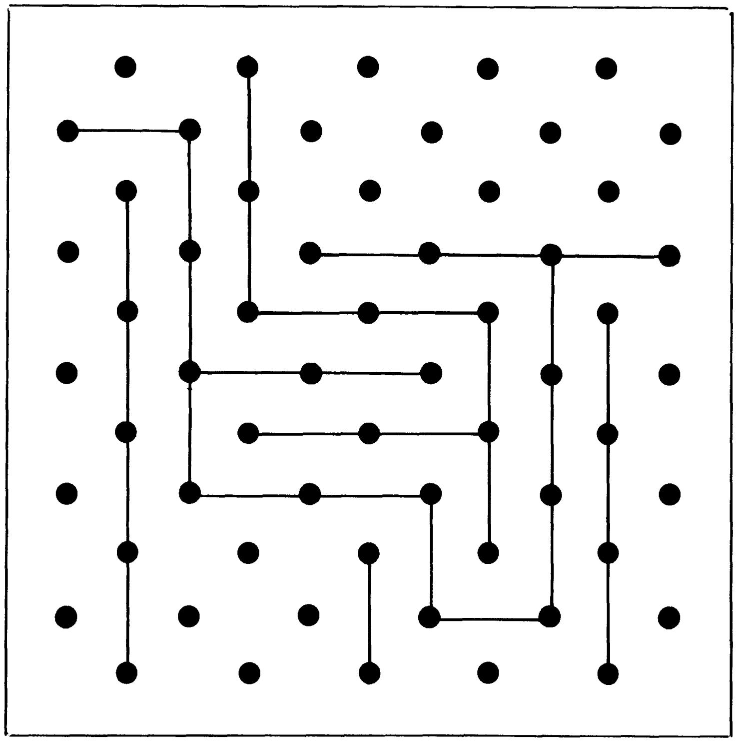

***FIG. 111***
***A finished game of Bridg-it. Red has won.***

learned of his discovery, I wrote immediately for details, expecting to receive a long, involved analysis that might prove too technical for this department. To my astonishment the explanation consisted of nothing more than the diagram reproduced in Figure 112 and the following two sentences: Make the first play as indicated by the black line at lower left in the diagram. Thereafter whenever your opponent's play crosses the end of a dotted line, play by crossing the other end of the same line. This ingenious pairing strategy guarantees a win for the first player, though not necessarily in the fewest moves. Gross describes his strategy as "democratic" in the sense that "it plays stupidly against a stupid opponent, shrewdly against a shrewd one, but wins regardless." This is not the only pairing strategy that Gross discovered, but he picked this one because of its regularity and the ease with which it can be extended to a Bridg-it board of any size.

Note that in the diagram no plays are indicated along the edges of the board. Such plays are allowed by the rules of Bridg-it (in fact, plays of this type are shown on the cover of the box), but there is no point in making such a move, because it can contribute nothing to winning the game. If in the course of playing the winning strategy your opponent throws away a play by making an

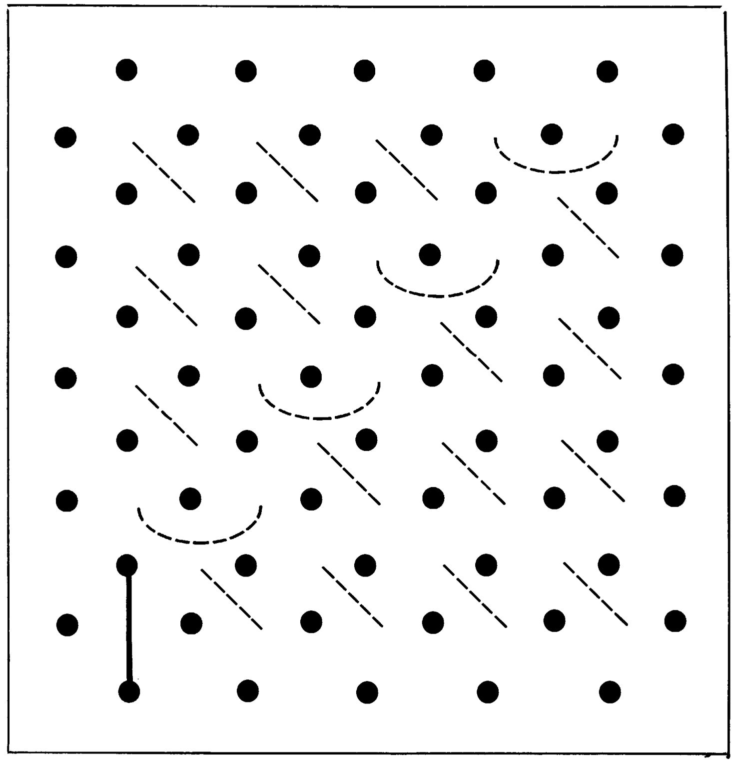

***FIG. 112***
***Oliver Gross's pairing strategy for winning at Bridg-it.***

edge move, you can counter with an edge move of your own. Or, if you prefer, you can play *anywhere* on the board. If at some point later in the game this random move is demanded by the strategy, you simply play somewhere else. Having an extra play on the board is sometimes an asset, never a liability. Of course, now that a winning strategy for the first player is known, Bridg-it ceases to be of interest except to players who have not yet heard the news.

Many board games with relatively simple rules have defied all attempts at mathematical analysis. An example is provided by the family of games that derives from halma, a game widely played in England late in the 19th century. "The normal English way," wrote George Bernard Shaw in 1898, is "to sit in separate families in separate rooms in separate houses, each person silently occupied with a book, a paper, or a game of halma. . . ." (This quotation is given in *The New Complete Hoyle,* by Albert H. Morehead, Richard L. Frey and Geoffrey Mott-Smith.)

The original halma (the name is a Greek word for "leap") was played on a checkerboard with sixteen squares to a side, but the basic mode of play was soon extended to other boards of varying size and shape. The game known today as Chinese checkers is one of the many later varieties of halma. I shall explain here only a simplified version, which can be played on the familiar eight-by-eight checkerboard and which leads to an entertaining solitaire puzzle that is still unsolved.

The game begins with the checkers in the standard starting position for a checker game. Moves are the same as in checkers, with these exceptions:

1. No jumped pieces are removed.
2. A checker may jump men of either color.
3. Backward moves and jumps are permitted.

A chain of unbroken jumps may be made over men of both colors, but one is not allowed to combine jumps with a nonjump move. The object of the game for each player is to occupy his opponent's starting position. The first to do so is the winner. A player also wins if the game reaches a situation in which his opponent is unable to move.

Some notion of how difficult it is to analyze games of the halma type can be had by working on the following puzzle. Arrange twelve checkers in the usual starting positions on the black squares of the first three rows of a checkerboard. The rest of the board is empty. In how few halma plays can you transport these men to the three rows on the opposite side of the board? A "play" is defined as either a diagonal checker move, forward or back, to a neighboring black square; or a jump over one or more men. An unbroken jump may include forward and backward leaps and is counted as a single play. As in halma, it is not compulsory to jump when jumps are available, and a series of unbroken jumps may be terminated wherever desired, even though more jumps are possible.

For convenience in recording a solution, number the black squares, left to right and top to bottom, from 1 to 32.

## ADDENDUM

AFTER THE twenty-move solution of the checker problem was published, several readers sent proofs that at least eighteen moves were required. One reader, Vern Poythress, Fresno, California, sent a twenty-move-minimum proof; unfortunately, too long and involved to give here.

As I pointed out in *The 2nd Scientific American Book of Mathematical Puzzles & Diversions,* Bridg-it is identical with a switching game called Bird Cage that was invented by Claude E. Shannon. The Shannon game is described in one of Arthur Clarke's short stories, "The Pacifist," reprinted in Clifton Fadiman's anthology, *Mathematical Magpie* (Simon and Schuster, 1962), pages 37–47; and in Marvin Minsky, "Steps Toward Artificial Intelligence," *Proceedings of the Institute of Radio Engineers,* Vol. 49, 1961, page 23. In addition to Bridg-it, manufactured by Hassenfeld Brothers, there is now a more complicated version of the game, using chess-knight-move connections, on the market under the name of Twixt, put out by 3M Brand Bookshelf Games.

Independently of Gross's work, a winning strategy for Bridg-it was discovered by Alfred Lehman, of the U.S. Army's Mathematical Research Center, University of Wisconsin. Lehman found a general strategy for a wide class of Shannon switching games, of which Bird Cage (or Bridg-it) is one species. Lehman wrote me that he first worked out his system in March 1959, and although it was mentioned in a Signal Corps report and in an outline sent to Shannon, it was not then published. In April 1961 he spoke about it at a meeting of the American Mathematical Association, a summary of his paper appearing in the association's June notices. A full, formal presentation, "A Solution of the Shannon Switching Game," was published in the *Journal of the Society of Industrial and Applied Mathematics,* Vol. 12, December 1964, pages 687–725. Lehman's strategy comes close to providing a winning strategy for Hex, a well-known topology game similar to Bridg-it, but Hex slipped through the analysis and remains unsolved.

In 1961 Günter Wenzel wrote a Bridg-it-playing program for the IBM 1401 computer, basing it on the Gross strategy. His description of the program was issued as a photocopied typescript by the IBM Systems Research Institute, New York City, and in 1963 was published in Germany in the March issue of *Bürotechnik und Automation.*

## ANSWERS

THE PROBLEM of moving twelve checkers from one side of the board to the other, using halma moves, brought a heavy response from readers. More than 30 readers solved the problem in 23 moves, 49 solved it in 22 moves, 31 in 21 moves and 14 in 20 moves. The fourteen winners, in the order their letters are dated, are: Edward J. Sheldon, Lexington, Massachusetts; Henry Laufer, New York City; Donald Vanderpool, Towanda, Pennsylvania; Corrado Böhm and Wolf Gross, Rome, Italy; Otis Shuart, Syracuse, New York; Thomas Storer, Melrose, Florida; Forrest Vorks, Seattle, Washington; Georgianna March, Madison, Wisconsin; James Burrows, Stanford, California; G. W. Logemann, New York City; John Stout, New York City; Robert Schmidt, State College, Pennsylvania; G. L. Lupfer, Solon, Ohio; and J. R. Bird, Toronto, Canada.

No proof that twenty is the minimum was received, although many readers indicated a simple way to prove that at least sixteen moves are required. At the start, eight checkers are on odd rows 1 and 3, four checkers on even row 2. At the finish, eight checkers are on even rows 6 and 8, four checkers on odd row 7. Clearly four checkers must change their parity from odd to even. This can be done only if each of the four makes at least one jump move and one slide move, thereby bringing the total of required moves to sixteen.

It is hard to conceive that the checkers could be transported in fewer than twenty moves, although I must confess that when I presented the problem I found it equally hard to conceive that it could be solved in as few as twenty moves. Assuming that the black squares are numbered 1 to 32, left to right and top to bottom, with a red square in the board's upper left corner, Sheldon's twenty-move solution (the first answer to be received) is as follows:

| | |
|---|---|
| 1. 21–17 | 11. 14–5 |
| 2. 30–14 | 12. 23–7 |
| 3. 25–9 | 13. 18–2 |
| 4. 29–25 | 14. 32–16 |
| 5. 25–18 | 15. 27–11 |
| 6. 22–6 | 16. 15–8 |
| 7. 17–1 | 17. 8–4 |
| 8. 31–15 | 18. 24–8 |
| 9. 26–10 | 19. 19–3 |
| 10. 28–19 | 20. 16–12 |

This solution is symmetrical. Figure 113 shows the position of the checkers after the tenth move. If the board is now inverted and the first ten moves are repeated in reverse order, the transfer is completed. So far as I know, this is the first published solution in twenty moves. It is far from unique. Other symmetrical twenty-move solutions were received, along with one wildly asymmetrical one from Mrs. March, the only woman reader to achieve the minimum.

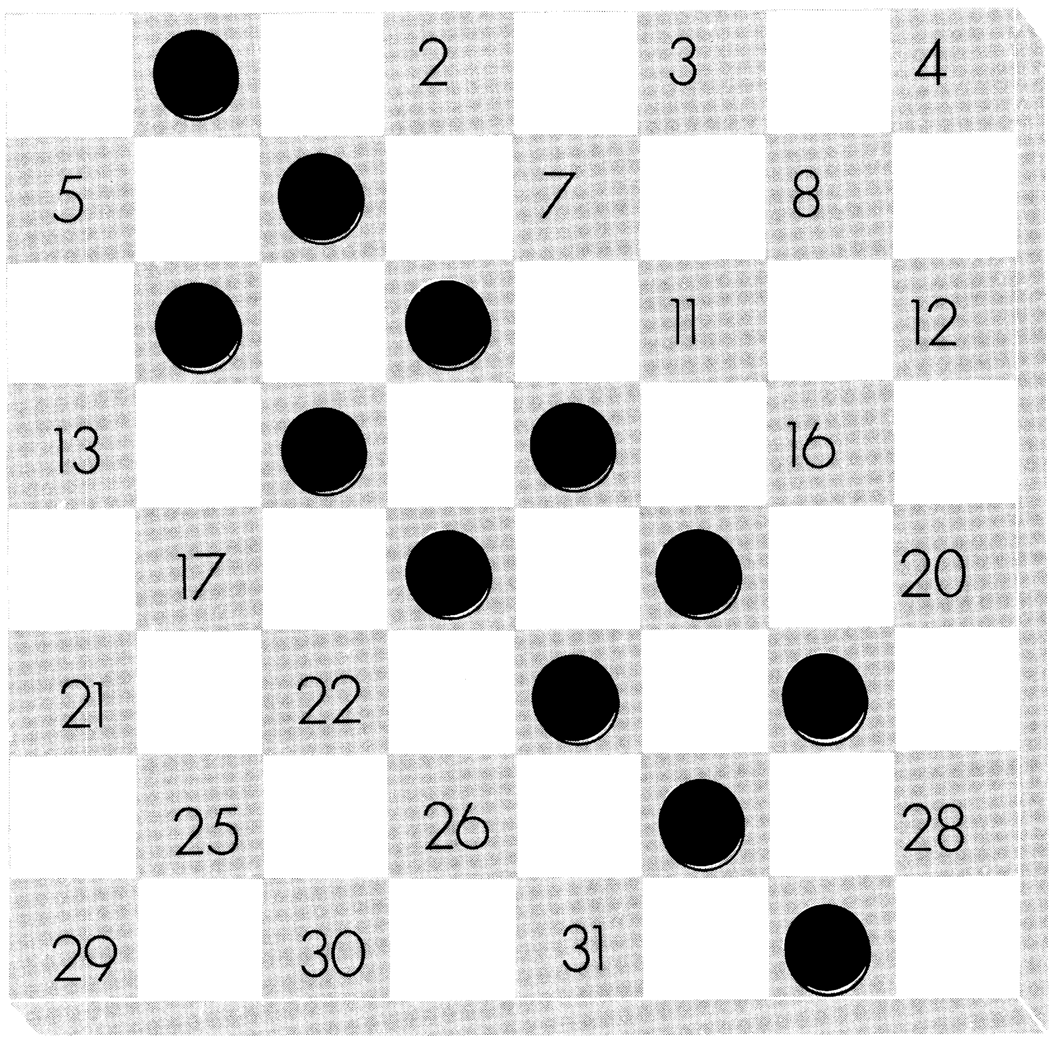


***FIG. 113***
***Position of checkers after ten moves.***