CHAPTER FIFTEEN

Nim and Tac Tix

ONE OF the oldest and most engaging of all two-person mathematical games is known today as Nim. Possibly Chinese in origin, it is sometimes played by children with bits of paper, and by adults with pennies on the counter of a bar. In the most popular version of the game 12 pennies are arranged in three horizontal rows as shown in Figure 86.

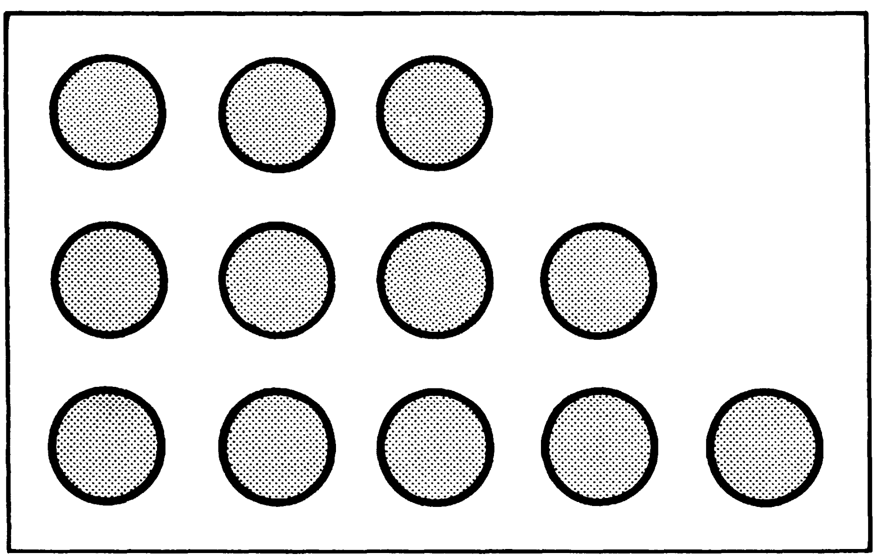

FIG. 86.
Twelve counters are arranged for a "3, 4, 5" game of Nim.

The rules are simple. The players alternate in removing one or more coins provided they all come from the same horizontal row. Whoever takes the last penny wins. The game can also be played in reverse: whoever takes the last penny loses. A good gamester soon discovers that in either form of the game he can always win if one of his moves leaves two rows with more than one penny in a row and the same number in each; or if the move leaves one penny in one row, two pennies in a second row and three in a third. The first player has a certain win if on his first move he takes two pennies from the top row and thereafter plays "rationally."

There is nothing startling about the foregoing analysis, but around the turn of the century an astonishing discovery was made about the game. It was found that it could be generalized to any number of rows with any number of counters in each, and that an absurdly simple strategy, using binary numbers, would enable anyone to play a perfect game. A full analysis and proof was first published in 1901 by Charles Leonard Bouton, associate professor of mathematics at Harvard University. It was Bouton, incidentally, who named the game Nim, presumably after the archaic English verb meaning to take away or steal.

In Bouton's terminology every combination of counters in the generalized game is either "safe" or "unsafe." If the position left by a player after his move guarantees a win for that player, the position is called safe. Otherwise it is unsafe. Thus in the "3, 4, 5" game previously described the first player leaves a safe position by taking two pennies from the top row. Every unsafe position can be made safe by a proper move. Every safe position is made unsafe by *any* move. To play rationally, therefore, a player must move so that every unsafe position left to him is changed to a safe position.

To determine whether a position is safe or unsafe, the numbers for each row are written in binary notation. If each column adds up to zero or an even number, then the position is safe. Otherwise it is not.

There is nothing mysterious about the binary notation. It is merely a way of writing numbers by sums of the powers of two. The chart of Figure 87 shows the binary equivalents of the numbers 1 through 20. You will note that each column, as you move from right to left, is headed by a successively higher power of two. Thus the binary number 10101 tells us to add 16 to 4 to 1, giving us 21 as its equivalent in the decimal system, based on the powers of 10. To apply the binary analysis to the 3, 4, 5 starting position of Nim, we first record the rows in binary notation as follows:

	4	2	1
3		1	1
4	1	0	0
5	1	0	1
Totals	2	1	2

	16	8	4	2	1
1					1
2				1	0
3				1	1
4			1	0	0
5			1	0	1
6			1	1	0
7			1	1	1
8		1	0	0	0
9		1	0	0	1
10		1	0	1	0
11		1	0	1	1
12		1	1	0	0
13		1	1	0	1
14		1	1	1	0
15		1	1	1	1
16	1	0	0	0	0
17	1	0	0	0	1
18	1	0	0	1	0
19	1	0	0	1	1
20	1	0	1	0	0

FIG. 87.
Table of binary numbers for playing Nim.

The middle column adds up to 1, an odd number, telling us that the combination is unsafe. It can therefore be made safe by the first player. He does so, as explained, by taking two pennies from the top row. This changes the top binary number to 1, thereby eliminating the odd number from the column totals. The reader will discover by trying other first moves that this is the only one which makes the position safe.

An easy way to analyze any position, provided there are no more than 31 counters in one row, is to use the fingers of your left hand as a binary computer. Suppose the game begins with rows of 7, 13, 24 and 30 counters. You are the first player. Is the position safe or unsafe? Extend all five fingers of your left hand, palm toward you. The thumb registers units in the 16 column; the index finger, those in the 8 column; the middle finger, the 4 column; the ring finger, the 2 column; the little finger, the 1 column. To feed 7 to your computer, first bend down the finger representing the largest power of 2 that will go into 7. It is 4, so you bend your middle finger. Continue adding powers of two, moving to the right across your hand, until the total is 7. This is of course reached by bending the middle, ring and little fingers. The remaining three numbers— 13, 24 and 30— are fed to your computer in exactly the same way except that any bent finger involved in a number is raised instead of lowered.

Regardless of how many rows there are in the game, if you finish this procedure with all your fingers raised, then the position is safe. This means that your move is sure to make it unsafe, and that you are certain to lose against any player who knows as much about Nim as you do. In this example, however, you finish with first and second fingers bent, telling you that the position is unsafe, and that you can win if you make a proper move. Because there are many more unsafe combinations than safe ones, the odds greatly favor the first player when the starting position is determined at random.

Now that you know that 7, 13, 24, 30 is unsafe, how do you find a move that will make it safe? This is difficult to do on your fingers, so it is best to write down the four binary numbers as follows:

	16	8	4	2	1
7			1	1	1
13		1	1	0	1
24	1	1	0	0	0
30	1	1	1	1	0
Totals	2	3	3	2	2

Note the column farthest to the left that adds up to an odd number. Any row with a unit in this column can be altered to make the position safe. Suppose you wish to remove a counter or counters from the second row. Change the first unit to 0, then adjust the remaining figures on the right so that no column will add up to an odd number. The only way to do this is to change the second binary number to 1. In other words, you remove all counters except one from the second row. The other two winning moves would be to take four from the third row or 12 from the last row.

It is helpful to remember that you can always win if you leave two rows with the same number of counters in each. From then on, simply move each time to keep the rows equal. This rule, as well as the preceding binary analysis, is for the normal game in which you win by taking the last counter. Happily only a trivial alteration is required to adopt this strategy to the reverse game. When the reverse game reaches a point (as it must) at which only one row has more than one counter, you must take either all or all but one counter from that row so as to leave an odd number of one-unit rows. Thus if the board shows 1, 1, 1, 3, you take all of the last row. If it shows 1, 1, 1, 1, 8, you take seven from the last row. This modification of strategy occurs only on your final move, when it is easy to see how to win.

Since digital computers operate on the binary system, it is not difficult to program such a computer to play a perfect game of Nim, or to build a special machine for this purpose. Edward U. Condon, the former director of the National Bureau of Standards who is now head of the physics department at Washington University in St. Louis, was a co-inventor of the first such machine. Patented in 1940 as the Nimatron, it was built by the Westinghouse Electric Corporation and exhibited in the Westinghouse building at the New York World's Fair. It played 100,000 games and won 90,000. Most of its defeats were administered by attendants demonstrating to skeptical spectators that the machine could be beaten.

In 1941 a vastly improved Nim-playing machine was designed by Raymond M. Redheffer, now assistant professor of mathematics at the University of California at Los Angeles. Redheffer's machine has the same capacity as Condon's (four rows with as many as seven counters in each), but where Nimatron weighed a ton and required costly relays, Redheffer's machine weighs five pounds and uses only four rotary switches. More recently a Nim-playing robot called Nimrod was exhibited at the Festival of Britain in 1951 and later at the Berlin Trade Fair. According to an account by A. M. Turing (in Chapter 25 of *Faster Than Thought*, edited by B. V. Bowden, 1953), the machine was so popular in Berlin that visitors "entirely ignored a bar at the far end of the room where free drinks were available, and it was necessary to call out special police to control the crowds. The machine became even more popular after it had defeated the economics minister, Dr. Erhard, in three games."

Among many variations of Nim which have been fully analyzed, one proposed in 1910 by the American mathematician Eliakim H. Moore is of special interest. The rules are the same as they are for regular Nim except that players are permitted to take from any number of rows not exceeding a designated number k. Surprisingly, the same binary analysis holds, provided a safe position is defined as one in which every column of the binary numbers totals a number evenly divisible by $(k + 1)$.

Other variations of Nim seem not to have any simple strategy for rational play. To my mind the most exciting of these as yet unanalyzed versions was invented about 10 years ago by Piet Hein of Copenhagen. (Hein is the inventor of Hex, a topological game discussed in Chapter 8.)

In Hein's version, called Tac Tix in English-speaking countries and Bulo in Denmark, the counters are arranged in square formation as shown in Figure 88. Players alter-

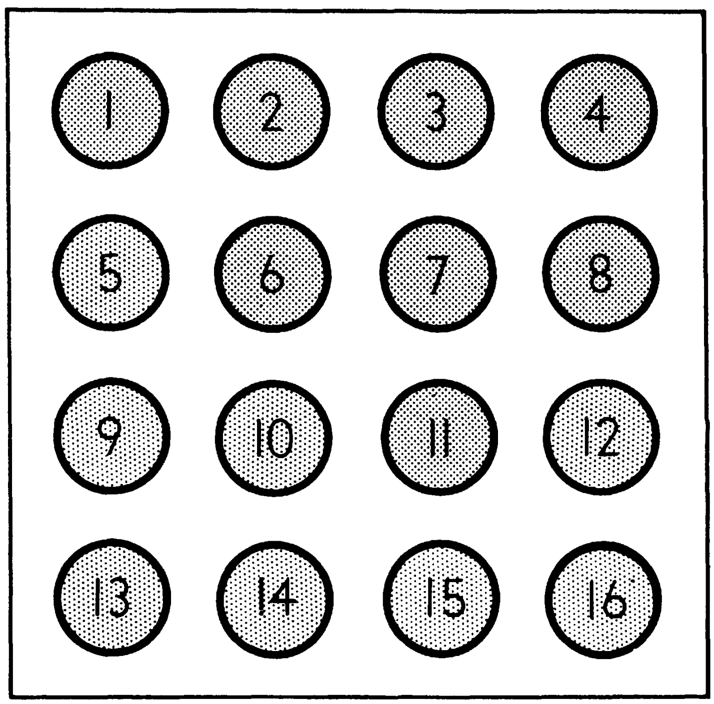

FIG. 88.
Piet Hein's game of Tac Tix.

nately take counters, but they may be removed from any horizontal or vertical row. They must always be adjoining counters with no gaps between them. For example, if the first player took the two middle counters in the top row, his opponent could not take the remaining counters in one move.

Tac Tix must be played in reverse form (the player who takes the last counter loses) because of a simple strategy which renders the normal game trivial. On squares with an odd number of counters on each side the first player wins by taking the center counter and then playing symmetrically opposite his opponent. On squares with an even number of counters on each side the second player wins by playing symmetrically from the outset. No comparable strategy is known for playing the reverse game, although it is not difficult to show that on a 3×3 board the first player can win by taking the center counter or a corner counter, or all of a central row or column.

The clever principle behind Tac Tix, that of intersecting sets of counters, has been applied by Hein to many other two- and three-dimensional configurations. The game can be played, for example, on triangular and hexagonal boards, or by placing the counters on the vertices and intersections of a pentagram or hexagram. Intersections of closed curves may also be used; here all counters lying on the same curve are regarded as being in the same "row." The square form, however, combines the simplest configuration with maximum strategic complexity. It is difficult enough to analyze even in the elementary 4×4 form, and of course as the squares increase in size the game's complexity rapidly accelerates.

A superficial analysis of the game suggests that symmetry play might insure a win for the second player in a 4×4 game, with only a trivial modification on his last move. Unfortunately, there are many situations in which symmetry play will not work. For example, consider the following typical game in which the second player adopts a symmetry strategy.

	FIRST PLAYER	SECOND PLAYER
1.	*5-6*	*11-12*
2.	*1*	*16*
3.	*4*	*13*
4.	*3-7 (wins)*	

In this example, the second player's initial move is a fatal one. After his opponent responds as indicated, the second player cannot force a win even if he departs from symmetry on all his succeeding moves.

The game is much more complex than it first appears. In fact it is not yet known whether the first or second player can force a win even on a 4 × 4 board from which the four corner pieces have been removed. As an introduction to the game, try solving the two Tac Tix problems (devised by Mr. Hein) which are pictured in Figure 89. On each board you are to find a move that insures a win. Perhaps some industrious reader can answer the more difficult question: Who has a win on the 4 × 4 board, the first or second player?

ADDENDUM

SEVILLE CHAPMAN, director of the physics division of the Cornell Aeronautical Laboratory, Inc., at Cornell University, sent me a wiring diagram for a well-thought-out portable Nim machine which he built in 1957. It weighs 34 ounces, using three multideck rotary switches to handle three rows of four to ten counters each. By taking the first move, the machine can always win. There is a rather pretty way to prove this. If we record the three rows in the matrix form previously described, it is clear that each row must have a "1" in either the 8 or 4 column, but not in both. (The two spaces cannot be empty, for then the number of count-

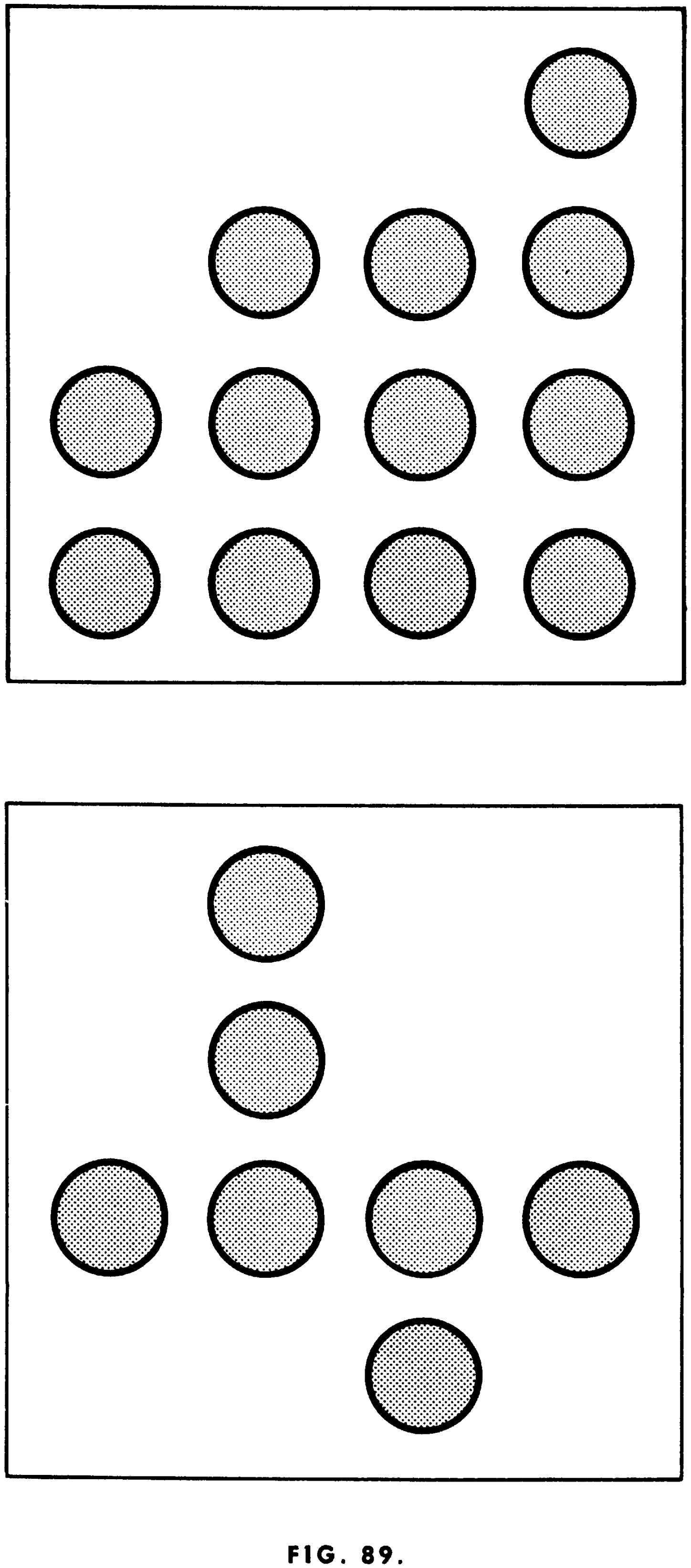

FIG. 89.
Two problems of Tac Tix.

ers in the row would be less than four, and they cannot both contain a "1" for then the number of counters would be more than ten.) There are only two ways that these three "1's" (one for each row) can be arranged in the two columns: all three in one column, or two in one column and one in the other. In both cases one column must total an odd number, making the initial position unsafe and thus guaranteeing a win for the machine if it plays first.

The following readers sent detailed analyses of the 4 × 4 Tac Tix game: Theodore Katsanis, Ralph Hinrichs, William Hall and C. D. Coltharp, Paul Darby, D. R. Horner, Alan McCoy, P. L. Rotherberg and A. A. Marks, Robert Caswell, Ralph Queen, Herman Gerber, Joe Greene, and Richard Dudley. No simple strategy was discovered, but there no longer is any doubt that the second player can always win on this board as well as on the 4 × 4 field with missing corner counters. It has been conjectured that on any square or rectangular board with at least one odd side, the first player can win by taking an entire center row on his first move, and that on fields with even sides the second player has the win. These conjectures are, however, not yet established by proofs.

As things now stand, the ideal board for expert Tac-Ticians who have mastered the 4 × 4 seems to be the 6 × 6. It is small enough to keep the game from being long and tiresome, yet complex enough to make for an exciting, unpredictable game.

ANSWERS

THE first Tac Tix problem can be won in several different ways: for example, take 9-10-11-12 or 4-8-12-16. The second problem is won by taking 9 or 10.