Connected Dominating Set and Short Cycles

Connected Dominating Set and Short Cycles

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Polynomial Kernels
No Polynomial Kernels, but FPT
W-hard

The impact of excluding short cycles.

Talk Outline

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Girth: at least seven

Talk Outline

Polynomial Kernels ($\geqslant 7$)

No Polynomial Kernels, but FPT

W-hard

Girth: five or six

Talk Outline

Polynomial Kernels $(\geqslant 7)$

No Polynomial Kernels, but FPT $(5, 6)$

W-hard

Girth: at most four

Talk Outline

Polynomial Kernels ($\geqslant 7$)
No Polynomial Kernels, but FPT ($5, 6$)
W-hard ($\leqslant 4$)

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The longer the cycles we exclude, the easier the problem becomes to solve.

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No Polynomial Kernels, but FPT $(5, 6)$
W-hard $(\leqslant 4)$

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Talk Outline

Part 1

Why Its Useful
to not have short cycles

Consider the problem of domination:

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Find a subset of at most k vertices S such that every $v \in V$ is either in S or has a neighbor in S.

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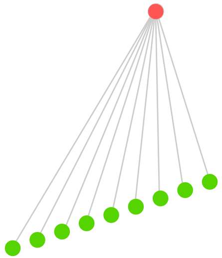
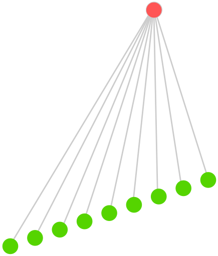
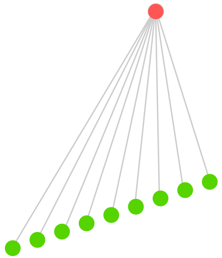
Let us consider graphs of girth at least five.
We begin by examining vertices of degree more than k.



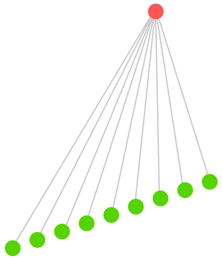
Does there exist a dominating set of size at most k that does *not* include the red vertex?



Who dominates the neighbors of the red vertex?
Does there exist a dominating set of size at most k that does *not* include the red vertex?

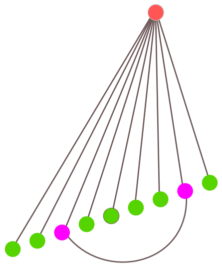


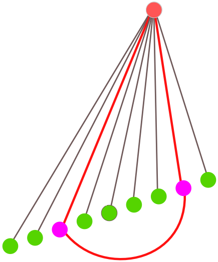
Who dominates the neighbors of the red vertex?
Clearly, it is not the case that each one dominates itself.



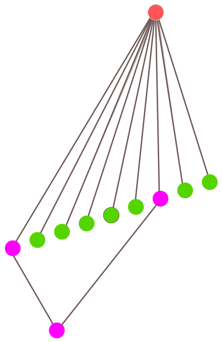
Who dominates the neighbors of the red vertex?
Clearly, it is not the case that each one dominates itself.

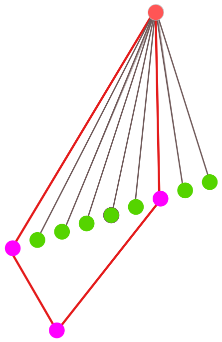
Either a vertex from within the neighborhood dominates at least two of vertices...





or a vertex from “outside” the neighborhood dominates at least two of the vertices…

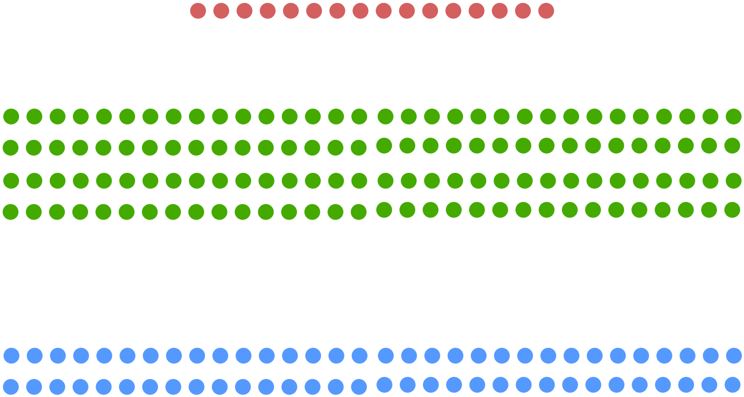




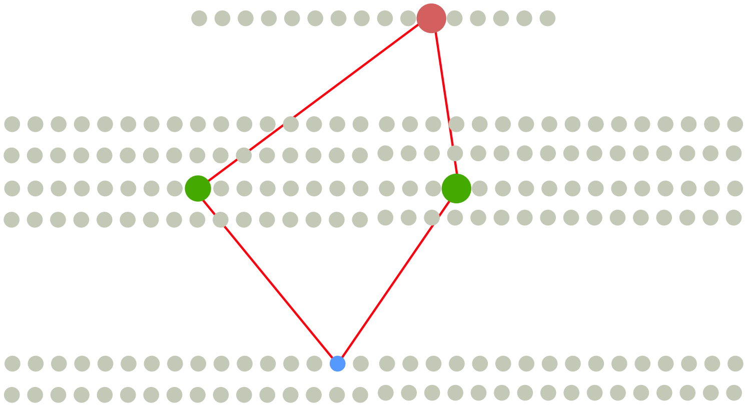
HIGH DEGREE VERTICES

On graphs that have girth at least five:

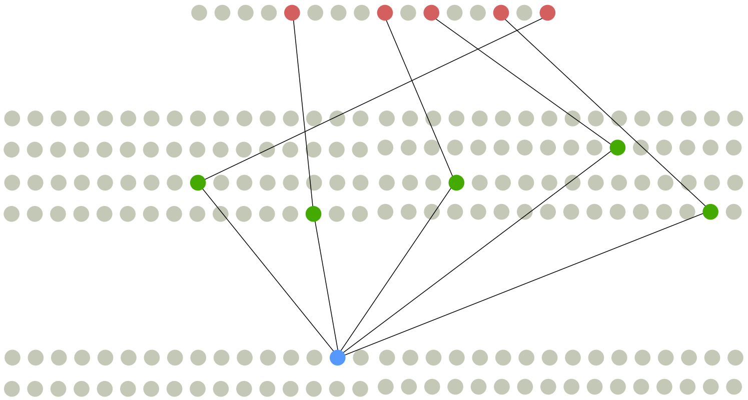
Any vertex of degree more than k belongs to any dominating set of size at most k.



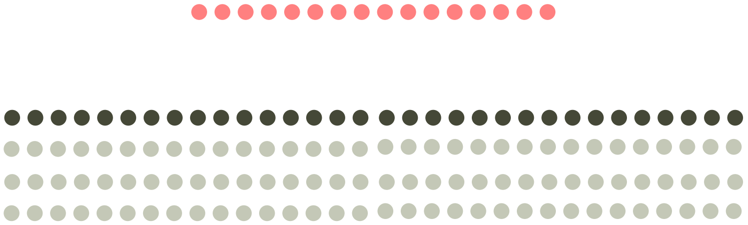
Notice that the number of reds is at most k and the number of blues is at most k^2.



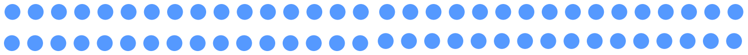
Greens in the neighborhood of a blue vertex cannot have a common red neighbor.



Thus, for every blue vertex, there are at most k green vertices.



vertices that have been dominated that have no edges into blue are irrelevant.



So now all is bounded, and we have a happy ending: a kernel on at most

$$O(k + k^2 + k^3)$$

vertices.

Part 2

FPT Algorithms
and Polynomial Kernels

In the connected dominating set situation, we have to backtrack a small way in the story so far to see what fails, and how badly.

HIGH DEGREE VERTICES

On graphs that have girth at least five:

Any vertex of degree more than k belongs to any dominating set of size at most k.

HIGH DEGREE VERTICES

On graphs that have girth at least five:

connected

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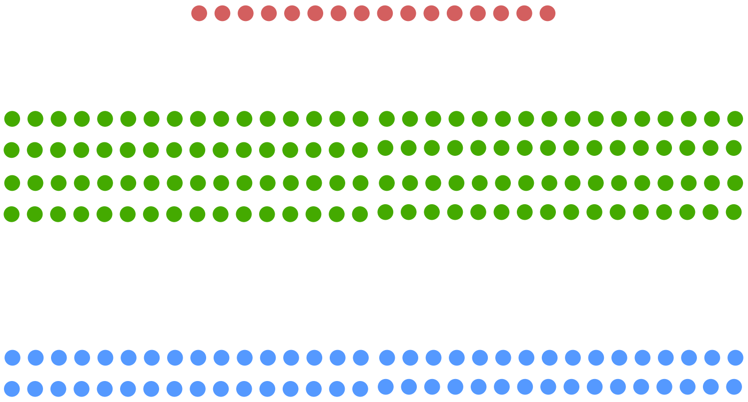
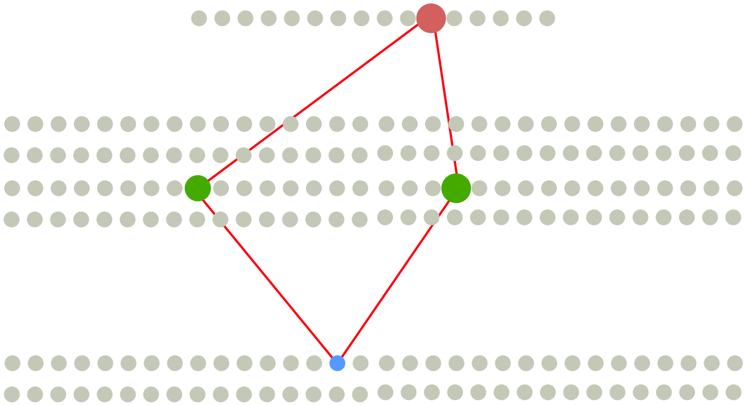


Notice that the number of reds is at most k and the number of blues is again at most k^2.



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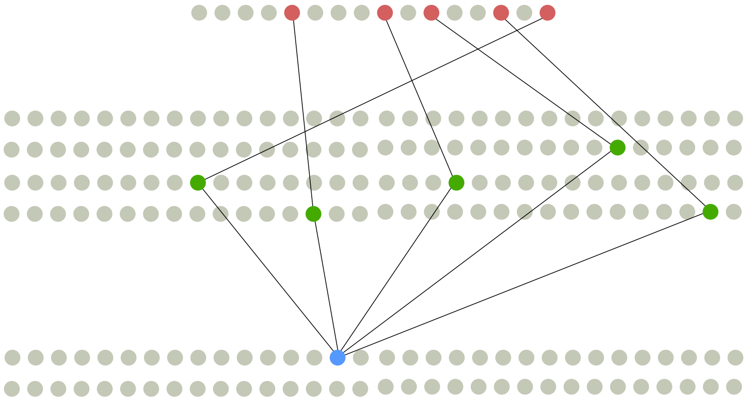

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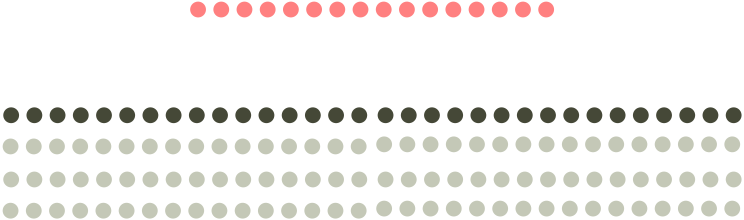

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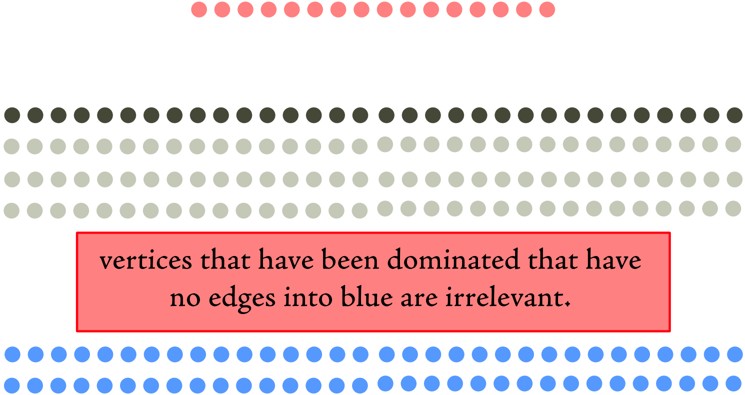
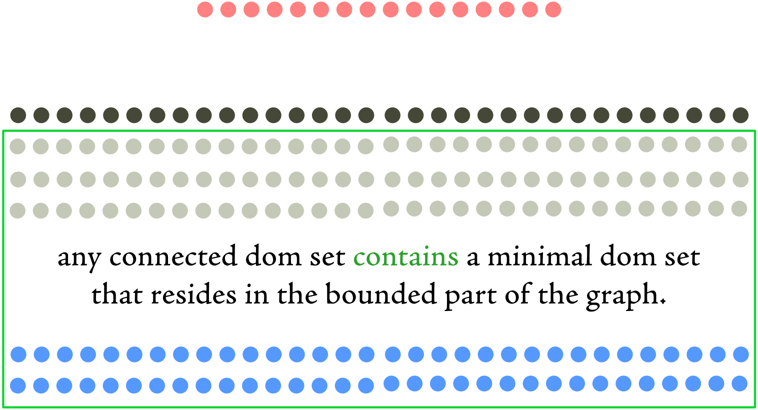

vertices that have been dominated that have no edges into blue are irrelevant.

Not any more...


any connected dom set contains a minimal dom set
that resides in the bounded part of the graph.

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Guess the minimal dominating set.

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Extend it to a connected dominating set by an application of a Steiner Tree algorithm.

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This is evidently FPT. What about kernels?

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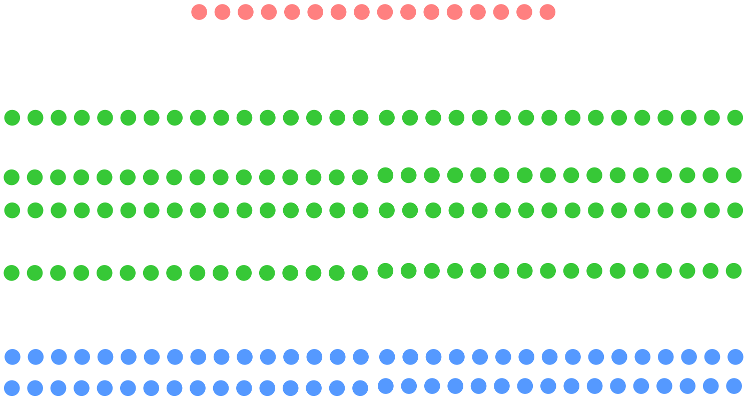
And we will show (later) that on graphs of girth five, connected dominating set is unlikely to admit polynomial kernels.

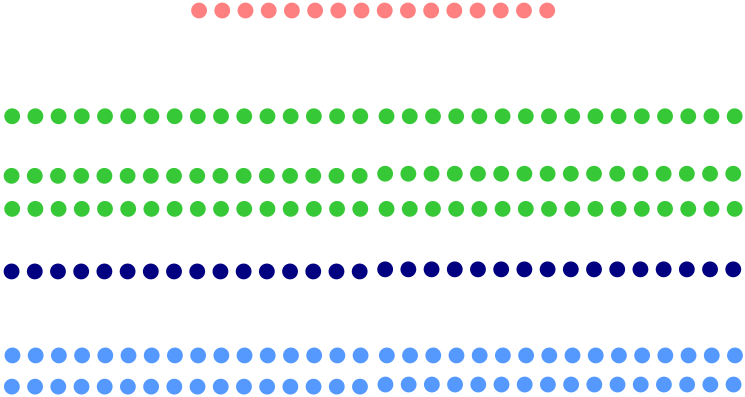
LOW DEGREE VERTICES

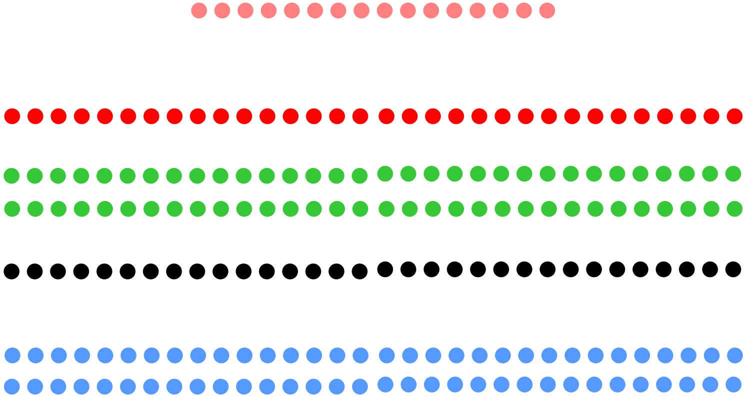
If there is blue pendant vertex, make its neighbor red.

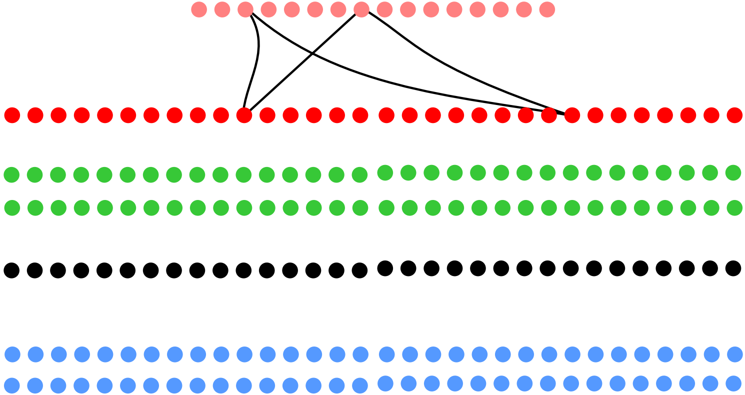
If there is a green vertex that is pendant, delete it.

If there is a blue vertex that is isolated, say no.

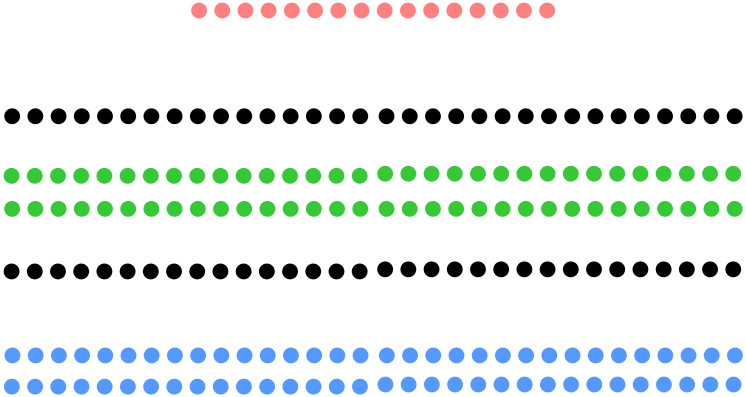


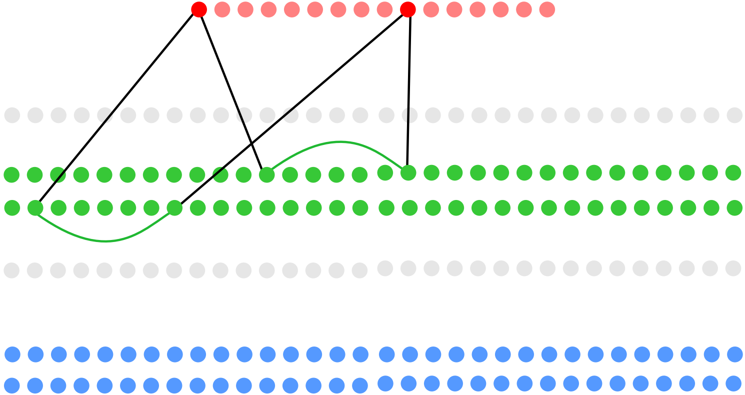






no pair 'sees' more than one vertex.





So now the green vertices are bounded:

- At most $O(k^3)$ with at least one neighbor in the blues,
- At most $O(k^2)$ with only red neighbors,
- At most $O(k^2)$ with no blue neighbors and at least one white neighbor.

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This gives us a $O(k^3)$ vertex kernel.

The Hard Part

Infeasibility of Polynomial Kernelization and W-hardness

Introducing...

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Fair Connected Colors

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FAIR CONNECTED COLORS

NP-complete ★ FPT ★ Compositional

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FAIR CONNECTED COLORS

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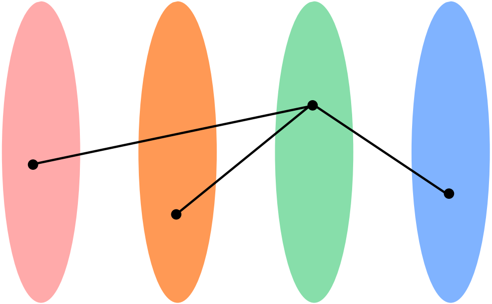
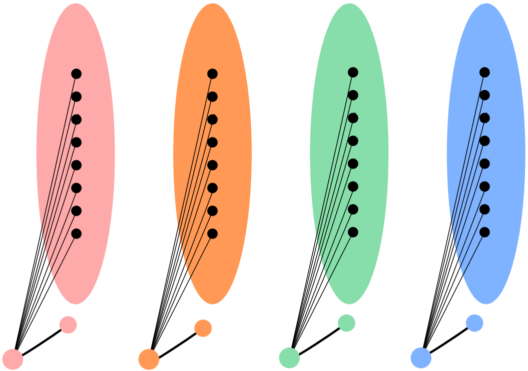
Ready for reduction!

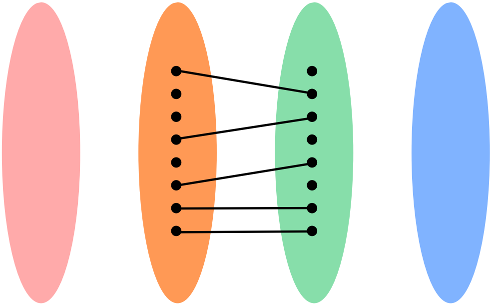


Promise: Every vertex has degree at most one into any color class, and every color class is independent.

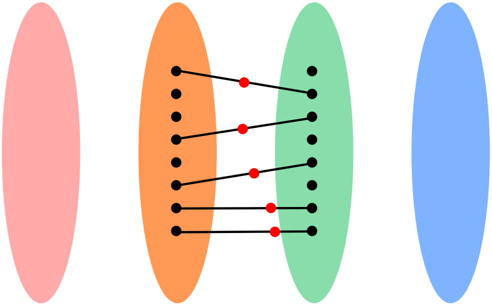
Question: Does there exist a colorful tree?



Add global vertices (with guards) to each of the color classes.

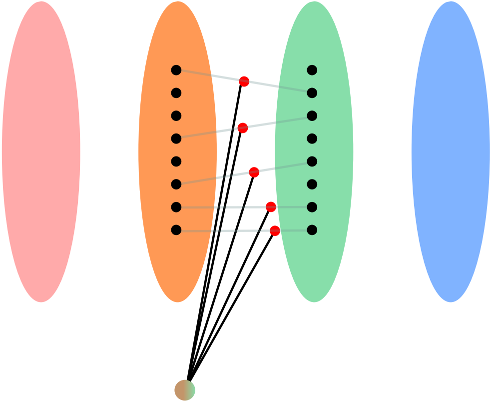


Between a pair of color classes:



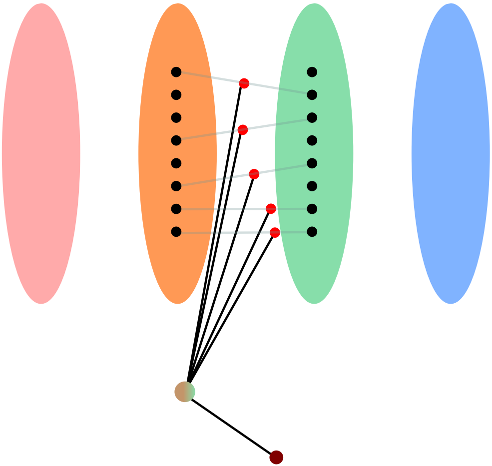
Between a pair of color classes:

- subdivide the edges,



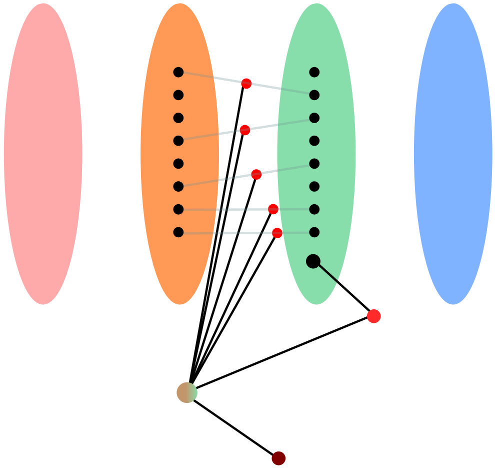
Between a pair of color classes:

- subdivide the edges,
- add a global vertex for the newly added vertices

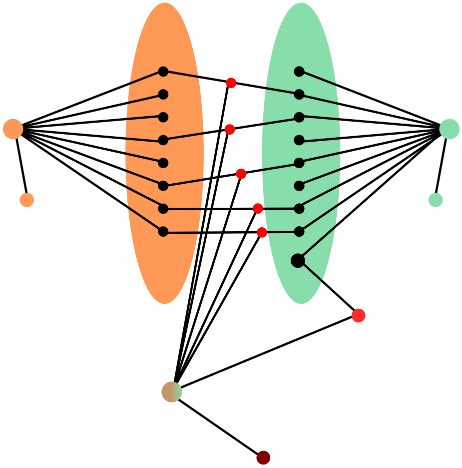


Between a pair of color classes:

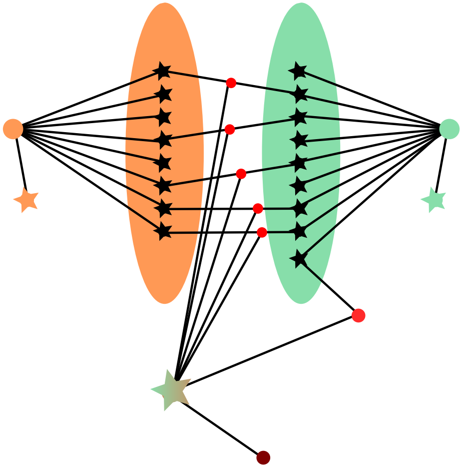
- subdivide the edges,
- add a global vertex for the newly added vertices (with guards).



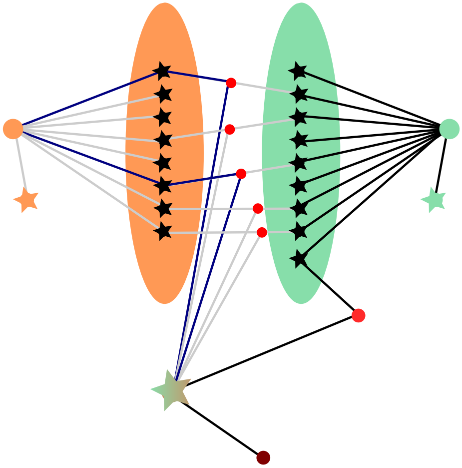
For vertices that don't look inside the neighboring color class, add a path of length two to the global vertex.



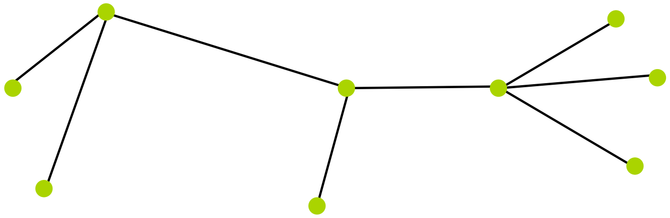
Here is $(1/k^2)^{th}$ of the full picture.



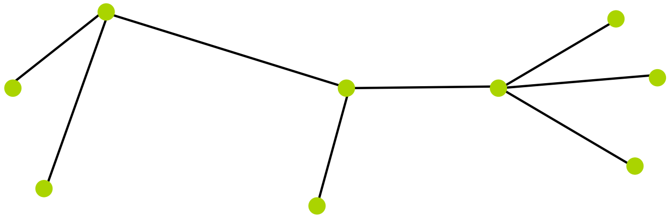
Notice that we have ended up with a bipartite graph...



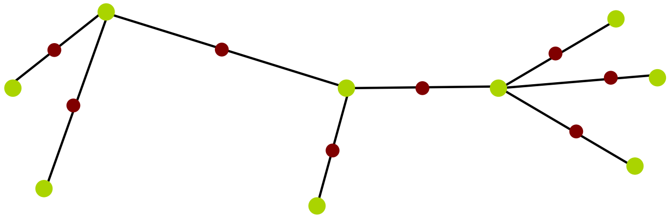
....with a cycle of length six.



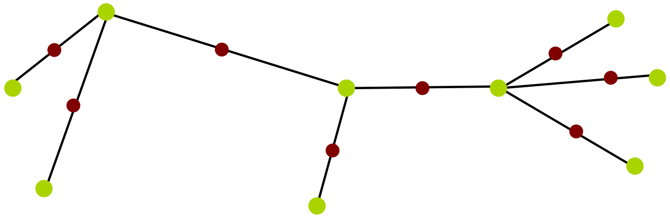
Suppose we have a colorful tree in the source instance.



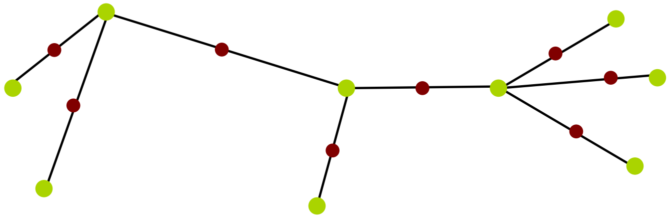
Consider the same subset.



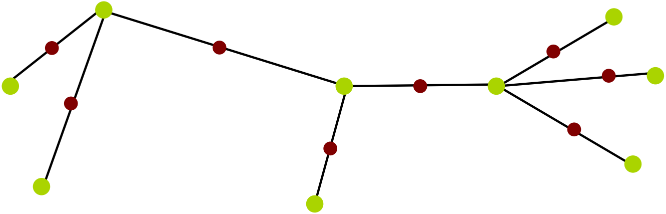
(along with the subdivided vertices along the edges of the tree)



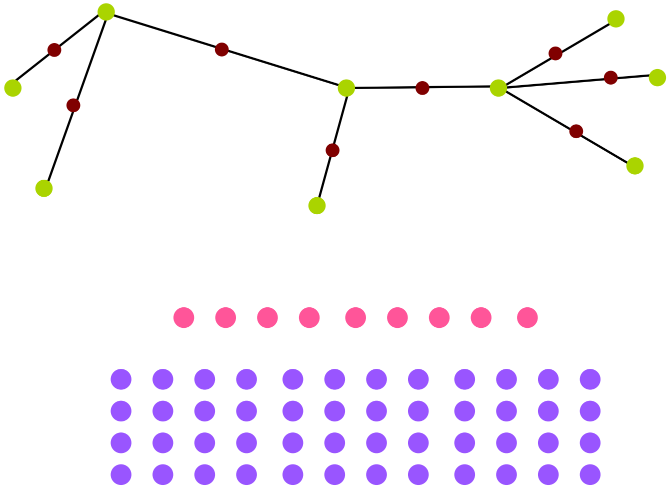
This is a connected subset, but not dominating.



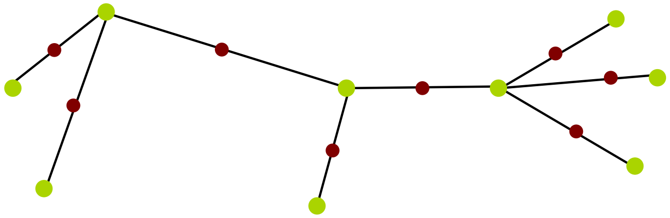
To fix this, add all the global vertices to this set.

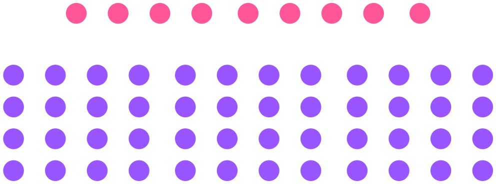


There are $k + \binom{k}{2}$ such vertices.

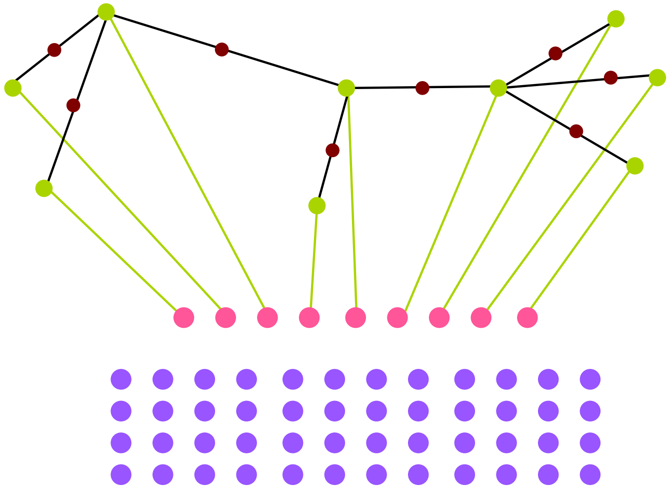


Now we have a dominating set, but it's not connected any more!

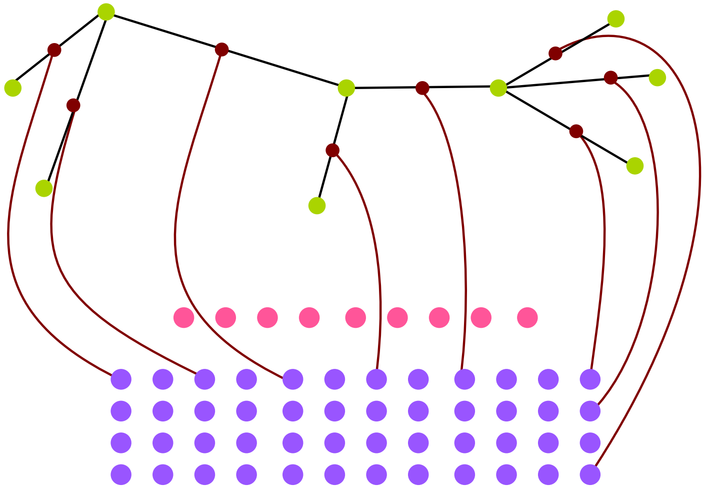




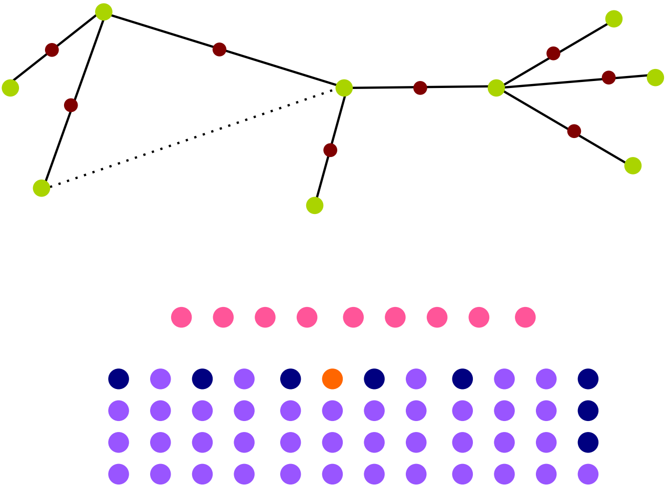
Let’s look more closely.



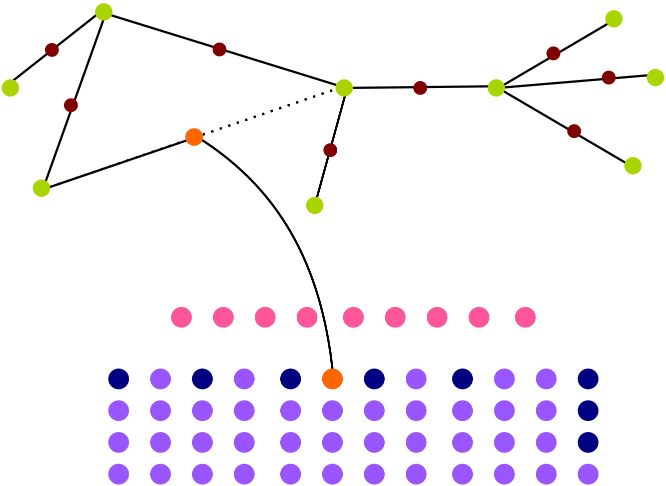
The vertices global to color classes are connected to the tree.



Some of the other vertices are connected too!



Each of the rest corresponds to a non-edge in the tree,



for which we add the path of length two to the picture.

The size of the connected dominating set thus obtained is:

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$$2\binom{k}{2} + 2k$$

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Two vertices for every pair of original vertices in the tree.
(A global vertex, and a neighbor.)

The size of the connected dominating set thus obtained is:

$$2\binom{k}{2} + 2k$$

Two vertices for every original vertex in the tree: itself, and the corresponding global vertex.

And that works out to...

$$2\binom{k}{2} + 2k$$

$$= k^2 + k$$

Let S be a connected dominating set of size at most:

$$\binom{k}{2} + \binom{k}{2} + k + k$$

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Of course, global vertices are forced in S, because of the guard vertices.

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Since S is a connected subset, each of these vertices have at least one neighbor in S.

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Because of the budget, each of them have...

Let S be a connected dominating set of size at most:

$$\binom{k}{2} + \binom{k}{2} + k + k$$

Since S is a connected subset, each of these vertices have *exactly one* neighbor in S.

Because of the budget, each of them have...

Thus, the dominating set picks exactly one vertex from each color class.

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- Neglect the global vertices and "subdivision vertices" of degree one, to be left with a connected subtree with subdivided edges.

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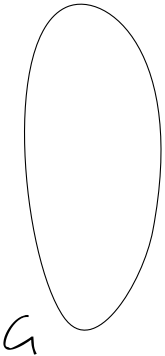
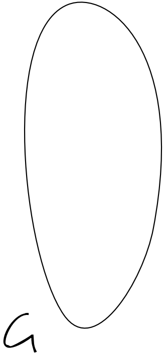
- Neglect the global vertices and "subdivision vertices" of degree one, to be left with a connected subtree with subdivided edges.
- This can be easily pulled back to a colorful tree of the original graph.

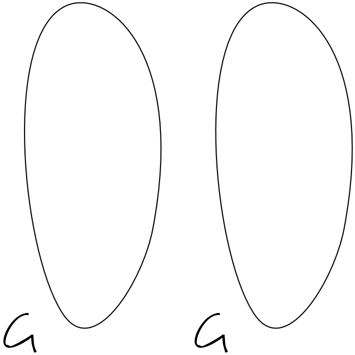
The W-hardness result.

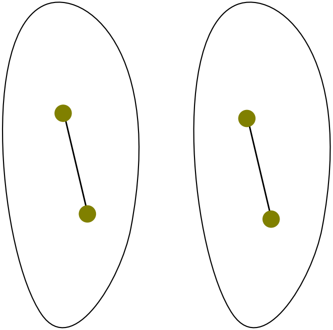


A dominating set instance

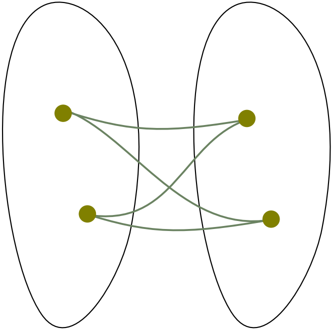


We begin by
making a copy of the graph.

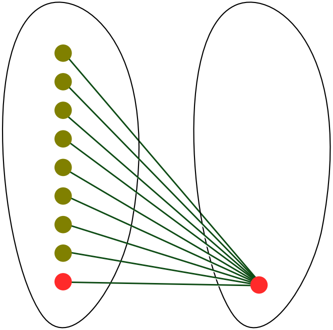

G
G



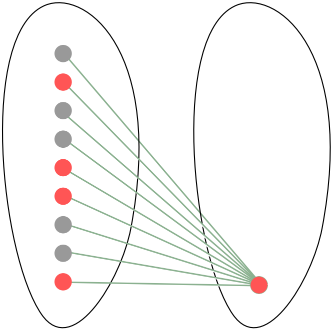
Then, for every edge in G,
we stretch it across the copies.



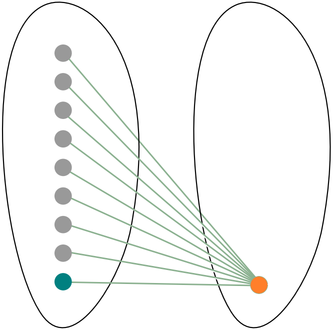
Then, for every edge in G,
we stretch it across the copies.



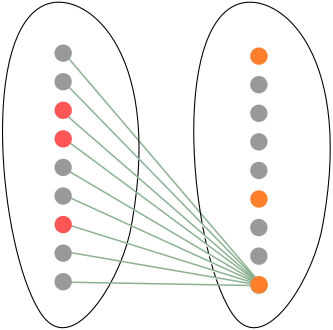
Finally, add a new vertex to each copy,
with one of them global to the other copy.



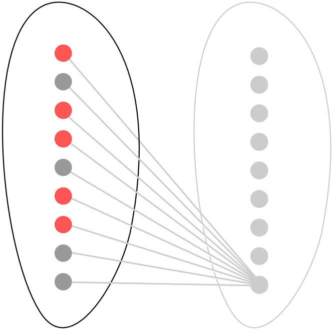
Any old dominating set + new global vertex
= new connected dominating set.



Any new dominating set contains the newly added global vertex WLOG.



Consider now the vertices of the new connected dominating set.



Observe that they form a dominating set of G.

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