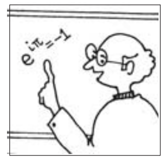

In this section of *Resonance,* we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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The Missing Boarding Pass

A counting problem that explores certain worries of the last passenger on an airplane.

1. Introduction

In this article, we deal with the problem of having a forgetful passenger on an airplane. The precise scenario is as follows:

On a plane that can accommodate one hundred passengers[1], the first passenger has lost his boarding pass. He chooses to sit at random. Every passenger thereafter seats himself in his own seat if he finds it, else he seats himself at random as well. What are the chances that the hundredth passenger is sitting on his own seat?

[1] Throughout the article, the passengers in the plane are assumed to be male, purely for reasons of convenience, and no bias towards either gender is intended.

To tackle the problem, we first make some normalizing assumptions:

1. The passengers enter the plane in a fixed order, which is arbitrary except for the fact that the first man is the man who has lost his boarding pass.

Keywords
Combinations, probability, one-one correspondence.

The important thing is that we will not allow two passengers to enter the plane at the same time. This means that when the ith person enters the plane, he should find $i-1$ seats occupied.

2. If a passenger finds that his seat is occupied, he will not fight or quarrel or make any fuss whatsoever about not getting his own seat, but will proceed to choose an empty seat at random like it is the most natural thing to do. (This is to ensure that our problem is solved with minimum delay.)

3. Once a passenger has seated himself, he cannot change his choice at any later stage, even if there are empty seats right in front of him or otherwise.

4. If a passenger's seat is empty when he enters the plane, it is absolutely certain that he will find it.

Appealing to intuition may not be the best way to approach the problem.

2. An Intuitive Approach

Appealing to intuition may not be the best way to approach the problem, particularly in the beginning, when any first attempt may look like an intimidating chaos of combinations (perhaps some of them even invalid; like trying to seat the first passenger in the third passenger's seat and the second passenger in the first passenger's). However, we will take a plunge anyway, before we mess around with the actual calculation.

Note that the 100th passenger does not really have a choice. He enters the plane, looks for *the* empty seat, sits down, period. He may just check, out of curiosity (or for the sake of our problem), if the seat is his designated or not.

It is perhaps natural to wonder whose seat this *could* be. We only want to consider the possibilities at the moment. If, for some especially absentminded group of

We claim that the 100th passenger can be occupying only one of two possible seats – either his own, or that of the first passenger.

passengers, everyone was forced to sit at random, then this 'leftover' seat could belong to any of the hundred people. We will return to this scenario later, let us tackle the present problem first.

We claim that the 100th passenger can be occupying only one of two possible seats – either his own, or that of the first passenger. To prove this claim, let us assume it is wrong to make such a claim. Then the 100th passenger may actually be seated on the seat of the ith passenger, where i is any number between 2 and 99. However, for the last man to occupy such a seat, it must be empty when he entered. This means that the seat in question must have been empty when its real owner had appeared earlier, due to axiom 3. But axiom 4 makes this situation a contradiction. Recall that whenever a passenger finds his seat empty, he is duty bound to occupy it. Therefore, it follows that the last passenger is either using his seat, or the first man's seat.

For an intuitive approach, that should be sufficiently interesting. Although this is a far cry from the answer we are looking for, it will aid in formulating the solution.

3. And the Chances Are...

3.1 *Working it Out*

We begin by introducing some more notation. We shall refer to passenger i by p_i. Let any 100-letter word formed using the alphabet p_i, $i \in I$ (where I is an index set of size 100) denote a possible seating arrangement. No letter may be repeated for obvious reasons. If the letter p_i is the jth letter of the word, then it is to be understood that p_i is occupying the jth passenger's seat.

Now, let U denote the set of all words which represent valid combinations, i.e., combinations that can be obtained without breaking any of the conditions outlined in Section 1. Now, let T denote the set of all words in

U whose concluding alphabet is p_{100}. In other words, we are collecting all combinations in which the last man does find his seat and calling the collection T. Also, let the complement of T with respect to U be denoted by F.

We define a mapping $\Gamma : T \to F$ as follows: for every word $t \in T$, let $\Gamma(t)$ be the word obtained by interchanging the positions of the first and hunderedth letter. The word thus obtained is clearly an element of F, and further, every element of F may be defined as $\Gamma(t)$ for some $t \in T$. Indeed, suppose this were not the case. Then there would exist at least one $f \in F$ which is not $\Gamma(t)$ for any $t \in T$. However, consider the word obtained when the first and hunderedth letters of f are interchanged. This has to be an element in T since T is the collection of all valid words for which the 100th letter is p_{100}. (Recall that all elements of F are words that *begin* with p_{100}, consistent with our reasoning that the last passenger either occupies the last seat or the first.) Thus we have found an element t in T so that $\Gamma(t)$ is f, contradicting our assumption that this was not possible. This implies that Γ describes a one-one correspondence between T and F, and it follows that both sets must have the same cardinality.

At this point, the reader may wonder if it is valid to assume that the function Γ, when applied to a word in T, always returns a 'valid' combination, in the sense of corresponding to a seating arrangement that respects axioms 1-4 described in Section 1. We make a couple of observations in this context.

Observation 3.1. It is always true that when passenger p_r enters the plane, the seats belonging to passengers $p_2, p_3, \ldots, p_{r-1}$ are occupied. This follows from the argument that if this were not the case, then some passenger has clearly violated axiom 4.

Lemma 3.1. *Suppose u is in T, and p_r is the first letter in u. Then, p_r is the last passenger to seat himself at random in the arrangement that corresponds to u, in other words, every passenger p_k, where $k > r$, is guaranteed to find his seat.*

Proof. This follows from the observation that a passenger $p_k, k > r$, who finds his seat occupied upon entry, will lead to the implication that there are k occupied seats, and hence k people on the plane excluding himself, which fails to respect axiom 1. This is because the first passengers' seat is taken (p_r), the kth passengers' seat is taken (since he claims to have found his seat occupied), and all intermediate seats are also occupied (as we observed in the previous remark) – giving us a total tally of k passengers excluding the kth. Since the mysterious new passenger cannot be explained, we conclude that everything that happens after the first passengers' seat is occupied is deterministic.

On the other hand, suppose u was in F, then we let p_r be the last letter in u. In this case as well, we claim that p_r is still the last passenger to seat himself at random. The argument is symmetric to the one provided for the case when u is in T. □

That the function is axiom-preserving follows now, since the two alphabets involved in our interchanging exercise are special ones – one is the last passenger, and the other is invariably the last person to seat himself at random. Thus, it is easy to see that when we interchange their positions, it cannot generate a combination that violates the rules.

Note, however, that we do not really require p_r to satisfy such a strong property for this mapping to be valid (the reader is encouraged to find a simpler proof, or at least, develop an intuition that does not rely on the observations above). We will soon see (in the more general case of this problem) that exchanges that do not necessarily involve passengers who are the last to seat themselves at random can also generate valid arrangements.

Having established the robustness of Γ, we proceed to our calculation. Begin by noticing that for every word u in U, we may associate with it a number $P\{u\}$, which denotes the probability of the occurence of u. For a given u, let R_u denote the ordered set of all passengers who seat themselves at random. Then the probability $P\{u\}$ is given by:

$$\prod_{i \in R_u} \frac{1}{(100 - i + 1)}$$

For instance, if the first passenger did find his seat, then the resulting word (where every passenger has occupied his own seat) will occur with probability 1/100.

However, we are not interested in the specific values of $P\{u\}$. Instead, consider an arbitrary but fixed word $t \in T$. We would now like to compare the probabilities of t and $\Gamma(t)$ – and the claim is that these probabilites are exactly equal:

Lemma 2.2. $P\{t\} = P\{\Gamma(t)\}$

Recall that

$$P\{t\} = \prod_{i \in R_t} \frac{1}{(100 - i + 1)}$$

and

$$P\{\Gamma(t)\} = \prod_{i \in R_{\Gamma(t)}} \frac{1}{(100 - i + 1)}.$$

To see that these products are equal, we only need to establish that $R_t = R_{\Gamma(t)}$, that is, the set of people who seat themselves at random is the same in t and $\Gamma(t)$. Let the first letter in t be p_k. Since the word $\Gamma(t)$ is the same as t but for the first and last letters, it suffices to observe that p_k is the last person to seat himself at random in both configurations, t, and $\Gamma(t)$. □

Recall that we split U into two mutually exclusive and exhaustive subsets T and F, and further established that these sets have equal cardinality. This implies that U itself has an even number of elements – let this number be $2h$. (Although the actual value of the number h is both unkown and irrelevant, the interested reader might refer to the appendix, where he will find a calculation that explicitly computes the size of U.) Every word in U represents an event, and the union of all events in U denotes the certain event, since we defined U to be the set of *all* valid words. Thus, the sum of probabilities of all these events is one.

$$\sum_{u \in T} P\{u\} + \sum_{u \in F} P\{u\} = 1$$

However, note that every word $u \in F$ can be written as $\Gamma(t)$ for some word $t \in T$, and since the mapping is bijective, we have

$$\sum_{u \in T} P\{u\} = \sum_{u \in F} P\{u\}$$

and it follows that

$$\sum_{u \in T} P\{u\} = 1/2$$

and this sum represents the union of all events in T, i.e., the event that the last passenger finds his seat.

It follows, therefore, that the chances of the 100th passenger finding his seat is exactly equal to the chances of getting a head when a fair coin is tossed, i.e., the chances are actually even!

3.2 *A Couple of Quick Remarks*

Two corollaries follow immediately from the result above. First, that the probability of the last passenger finding his own seat is independent of the capacity of the plane. The generalization of the argument in Section 3.1 to a plane with n seats instead of hundred is fairly straightforward, and the reader who looked at the Appendix should be able to see that the number of combinations for which the last passenger finds his seat is 2^{n-2} and the remaining combinations also amount to 2^{n-2}. The result, therefore, holds for planes built for a couple as well as imaginary planes (of finite size) as big as the universe.

The second corollary modifies assumption (1) of Section 1 a little. We consider the case where the passenger who has lost his boarding pass *doesn't enter first*. So we consider the case where p_k ($k \neq 100$ for obvious reasons) is the passenger who has lost his pass, and therefore will be the first person to seat himself at random. In this situation, obviously, the first k passengers will occupy their own seats with probability 1. Therefore we may as well be looking at a new, smaller plane that has $n - k$ seats whose first passenger has to seat himself at random. Given that this situation is equivalent to having the forgetful passenger being p_k in the context of a plane with n seats, the problem is now trivial – we are now back to our original problem, except that we are dealing with $n-k$ seats instead of n. And since we have already seen that this leaves our original result unchanged, we now have the following:

The chances of the last passenger finding his own seat remains one-half, irrespective of when the passenger who has misplaced his boarding pass enters, i.e., the probability is independent of the timing of the first random choice.

Conclusion 3.1

The chances of the last passenger finding his own seat remains one-half, irrespective of when the passenger who has misplaced his boarding pass enters, i.e., the probability is independent of the timing of the first random choice.

4. One Step Further – a Generalization

At this stage, it is perhaps natural to ask how the chances of our last passenger change, if the number of people who *have* to seat themselves at random increases from one to two or some other arbitrary number less than hundred, or, for that matter, less than n (now that we know that there is nothing particularly divine about the number hundred, we will use a plane of capacity n henceforth). Suppose that the first k passengers have misplaced their tickets. (If this sounds too unreal, then assume k arbitrary passengers have lost their tickets and then the airline management asked them to go in first for their – and our – convienence.) We will find that the solution to this problem is only a natural generalization of the solution to the problem concerning the single passenger who lost his pass.

We use the same notation as we did earlier, namely, that any n-letter word formed using the alphabet p_i, $i \in I$ (where I is an index set of size n) denotes a possible seating arrangement. No letter may be repeated for obvious reasons. If the letter p_i is the jth letter of the word, then it is to be understood that p_i is occupying the jth passenger's seat.

Again, let U denote the set of all words which represent valid combinations, i.e., combinations that can be obtained without breaking any of the conditions outlined in Section 1. Now, let T denote the set of all words in U whose concluding alphabet is p_n. In other words, we are collecting all combinations in which the last man does find his seat and calling the collection T. We also observe that when the last man is not occupying his own seat, he is occupying one of the first k seats. This has intuitive appeal, since if this were not the case, we immediately contradict axiom 4, just as we did in the simpler version of the problem. This motivates the definiton of k other subsets of U as follows: in the subset F_i, collect all words in U that have p_n for the ith alphabet.

We define k mappings $\Gamma_i : T \to F_i$ as follows: for every word $t \in T$, let $\Gamma_i(t)$ be the word obtained by interchanging the positions of the ith letter with the last. The word thus obtained is clearly an element of F_i, and further, every element of F_i may be defined as $\Gamma_i(t)$ for some $t \in T$. To prove this, again, suppose this were not the case. Then there would exist at least one $f \in F_i$ which is not $\Gamma_i(t)$ for any $t \in T$. However, consider the word obtained when the ith and nth letters of f are interchanged. This has to be an element in T since T is the collection of all valid words for which the nth letter is p_n. (Recall that all elements of f are words for which the ith letter is p_n, by definition.) Thus we have found an element t in T so that $\Gamma_i(t)$ is f, contradicting our assumption that this was not possible. This implies that Γ_i describes a one-one correspondence between T and F_i, and it follows that both sets must have the same cardinality. Since this is true for every $i \in \{1, 2, \ldots k\}$, it implies that all the F_i's are also of equal size.

The question of whether any of the Γ_is will actually produce a valid mapping also arises, and may be dealt with as follows:

Note that the word $\Gamma_i(t)$ is a valid seating arrangement if we can be assured that the passenger who was at p_i's seat in the word t was someone who sat at random. If this is true, then he could have chosen the nth passengers' seat instead of the ith passengers', since that is guaranteed to be empty (remember that t is in T, where the nth passenger always finds his seat, and therefore his seat is always empty when a passenger before him enters the plane). Subsequently, everyone else seats themselves as they would have done in t (no one would be affected by p_i's change of mind, since in t, no one would have taken the nth seat anyway), and finally the nth passenger gets p_i's seat which is necessarily the only one left. Note that this is exactly the description of $\Gamma_i(t)$.

So we only need to observe that the passenger who was at p_i's seat in the word t was someone who sat at random. But this is true because if this passenger is not p_i himself, then he's out of his own seat, and is therefore sitting at random. If the passenger is p_i, although he is someone who is occupying his designated seat, observe that this is purely a happy accident – since i is less than k, p_i doesn't have a boarding pass and happens to be (in this case) making a correct but random choice.

Again, for every word u in U, we may associate with it a number $P\{u\}$, which denotes the probability of the occurence of u. For a given u, let R_u denote the ordered set of all passengers who seat themselves at random. Then the probability $P\{u\}$ is given by:

$$\prod_{i \in R_u} \frac{1}{(n-i+1)}.$$

However, again, we are not interested in the specific values of $P\{u\}$. Instead, consider an arbitrary but fixed word $t \in T$. Let us compare the probabilities of t and $\Gamma_i(t)$ for an arbitrary but fixed i. The claim here is analogous to Lemma 3.1, and we have, predictably:

Lemma 4.1 $P\{t\} = P\{\Gamma_i(t)\}$

Proof. Again, we only need to show that the set of people who seat themselves at random in t and $\Gamma_i(t)$, for any fixed but arbitrary i, are the same. Equivalently, we would like to be sure that people who haven't lost their boarding pass and have found their seats empty in t, continue to find them in $\Gamma_i(t)$. Note that the only passengers involved in the exchange are the passengers who happen to be occupying p_i's seat in t (call him p_r), and the hunderedth. The nth passenger walks in last, so he cannot affect any choices before him. p_r occupies the nth passengers' seat instead of p_i's, and we observe that no choice for a word in T can involve this seat. So the only person robbed of his seat is the last passenger, who does not make a choice anyway. This concludes the proof. □

Now, suppose the cardinality of U is $(k+1)h$. (The cardinality is a multiple of $(k+1)$ since, by defining subsets T and $F_i, i = 1, 2, \ldots, k$, we effectively partitioned U into $(k+1)$ disjoint subsets of equal sizes). Every word in U represents an event, and the union of all events in U denotes the certain event, since we defined U to be the set of *all* valid words. Thus the sum of probabilities of all these events is one.

$$\sum_{u \in T} P\{u\} + \sum_{j=1}^{k} \sum_{u \in F_j} P\{u\} = 1.$$

However, note that every word $u \in F$ can be written as $\Gamma_i(t)$ for any i and some word $t \in T$; and since the mapping is bijective, we have

$$\sum_{u \in T} P\{u\} = \sum_{u \in F_i} P\{u\},$$

for any i – and it follows that:

$$\sum_{u \in T} P\{u\} = 1/(k+1)$$

and this sum represents the union of all events in T, i.e., the event that the last passenger finds his seat.

Thus we have actually proved the following:

THEOREM 4.1 *The probability that the last passenger finds his seat when the first k passengers seat themselves at random is* $1/(k+1)$.

The probability that the last passenger finds his seat when the first k passengers seat themselves at random is $1/(k+1)$.

Now let us explore another possibility, that of getting a computer do the work. Trying to get your computer to output *one specific valid seating arrangement* is not very difficult.

[2] $[n] = \{1, 2, \ldots, n\}$

[3] This is done in the order that the passengers arrive, so for the purposes of the present discussion we agree that passenger p_i arrives at time i.

A particular seating arrangement may be viewed as a permutation of the set $[n]$ [2]. The algorithm will attempt to assign people to seats[3], either at random or otherwise, depending on whether the seat designated to the passenger in question is empty or not. To begin with, no assignments are made, and the first assignment is made at random with each seat having an equal chance of being assigned to p_1. At any given point of time, the algorithm keeps track of all the seats which are filled. This includes (recall from Observation 3.1) seats between 2 and $i-1$. It also either includes the seat belonging to p_i, or it does not – in case it is the latter, the algorithm simply associates p_i with his rightful seat, and if not, then a random assignment is made. Also note that if, at any point of time, a random assignment involves passenger p_i being given p_1's seat, then the algorithm terminates after giving all passengers $p_j, j > i$ their own seats.

1. Let $x, i, j, r, k, A[n]$ be local variables.
2. Initialize x to 0 and all elements of $A[i]$ to -1.
3. Get values of n, k from the user.
4. Repeat steps $5 - 8$ while $x < n$.
5. If $\{x = 0\}$, then:
   (a) Store $\Pi(n)$ in r. [4]
   (b) $A[r] = x$
6. Else if $\{(A[x] = -1) \bigcap x > k\}$, then:
   A[x]=x;*(s_x, p_x)*
7. Else:
   (a) $r = \Pi(n - x)$
   (b) $j = 0$
   (c) for$\{i = 0, i < n, i = i + 1\}$

[4] Here we assume, without loss of generality, that $\Pi(x)$ generates a random number between 0 and x–1, instead of generating one between 1 and x.

- if$\{(A[i] = -1) \bigcap (j <= r)\}$ j=j+1.
- else if$\{j = r\}$, then skip loop and go to (d).
- else return to loop (c).

(d) $A[i] = x$

8. $x = x + 1$
9. Increment all array elements and array indices by one, and print both sets of values.

In the algorithm above, the contents of the array A correspond to various p_i's, i.e., the ith element of A corresponds to the person occupying s_i, the ith passenger's seat. Step 6 of the algorithm corresponds to the situation where p_x walks in to find s_x unoccupied. In the first case under the loop 7(c), $A[i] = -1$ indicates an empty seat, and since we are looking for the $\Pi(n-x)$ th empty seat, we increment j only as long as *both* conditions hold. In the second case, the rth empty seat has been found, so we terminate the search there. The rest of the algorithm is self-explanatory.

This algorithm is going to give us one among the possible T_k combinations corresponding to (n, k). We make a couple of remarks before we conclude the discussion of simulating the problem on a computer:

Remark 4.1 The algorithm above can be modified so that instead of generating random numbers using an in-built library function, the user can enter a seat number of his choice. We omit the details since the modification is trivial. Do not forget to account for the user entering, inadvertently or otherwise, a number that does not lie in the valid range!

Remark 4.2 Running the algorithm repeatedly may not give an accurate approximation of the ratio that was proposed in the conjecture. These statistics should not be used for drawing conclusions because they depend on how $\Pi(x)$ is designed, and may not always lead to the proposed probability.

5. A Different Seat Game

We pose one last set of questions on the scenario of the first k passengers losing their boarding passes:

1. What are the chances of the last k passengers finding their own seats if $k < \lfloor n/2 \rfloor$?

2. What are the chances of the last $n-k$ passengers finding their own seats if $k > \lfloor n/2 \rfloor$?

The second question is much less exciting than it might look. The k passengers can leave $n-k$ empty seats in $\binom{n}{n-k}$ ways. Precisely *one* of these combinations, $\{k+1, k+2, \ldots, n\}$, will give us the required situation, viz., the last $(n-k)$ passengers occupying their own seats. Thus the required probability is given by $1/\binom{n}{n-k}$, which amounts to:

$$k!/\prod_{i=k-1}^{0}(n+i).$$

The first question, however, is not so straightforward. This is because when $k < \lfloor n/2 \rfloor$, there will be a few passengers who come after the first k and before the last $n-k$. In the second question, the set of these passengers is a null set, which simplifies matters a great deal. Now, this non-empty set will create a chaos of new combinations, all of which need to be accounted for to get to the correct answer. One more time, the reader is welcome to give the problem a try.

Also, what if the k passengers who lost their boarding passes refuse to line up, so that, instead of the *first k passengers*, we are dealing with k arbitrary passengers having lost their passes? Would the last passenger's chances be affected? How would the algorithm change?

It is now finally time to say good bye to the friendly neighborhood airport, and the notorious plane. The problem of the misplaced boarding passes can be extended a great deal, as indicated above. We may also wonder:

1. What is the probability that, five minutes after takeoff, the last passenger discovers he's boarded the wrong plane?

2. What is the probability that the plane will run out of fuel mid-air?

3. Where did all the boarding passes go?

We leave the answers to the reader's imagination. May you be blessed with many amnesiac co-passengers.

6. Acknowledgements

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Suggested Reading

[1] W Feller, *Probability Theory*, Willy Eastern Edition, 1978.

[2] Papoulis, *Probability, Random Variables and Stochastic Processes*, McGraw Hill Companies, Fourth Edition, 2002.

[3] *A Walk Through Combinatorics*, Miklós Bóna, First Edition, World Scientific, 2002.

Appendix

In this appendix, we get down to pen and paper mode to calculate the number of possible seating arrangements in a 100-seater plane, i.e., the case of $n = 100$ and $k = 1$.

We begin by introducing some notation. Let s_i refer to seat number i, and, along the same lines, let p_j refer to the jth passenger. Also, let $N(s_i, p_j)$ denote the number of possible arrangements allowed by the axioms for a fixed i-j pair, of course, $i \in \{1, 2, 3, \ldots, 100\}$ and , $j \in \{1, 2, 3, \ldots, 100\}$. This corresponds to the number of ways in which the 'remaining non-j' passengers can choose seats for themselves given that p_j has occupied s_i. It is important to see that evaluating this expression may not always be a straightforward exercise. For instance, $N(s_1, p_1)$ is clearly 1, but it is not so easy to determine $N(s_{66}, p_2)$.

Our aim will be to make clever selections of pairs of numbers, (i, j) so that we accomplish two things at the same time - first, we account for *all possibilities*, and second, we ensure that the computation of the sequence of numbers $N(s_i, p_j)$ does not get too involved.

We have, trivially, $N(s_1, p_1) = 1$ and $N(s_2, p_1) = 1$. Also, the reader should be able to verify that we exhaust all possible seating arrangements when we consider nn

$$N(s_1, p_i), i \in \{1, 2, \ldots, 100.\}$$

and it follows that the total number of seating arrangements is given by:

$$\sum_{i=1}^{100} N(s_1, p_i).$$

To evaluate $N(s_1, p_3)$, we need to observe that p_3 will occupy s_1, i.e., a seat other than his own iff s_3, his own seat, is occupied. Due to assumption (1) in Section 1, we see that only two people can occupy s_3, namely p_1 and p_2. Again, p_2 will occupy s_3, a seat other than his own, iff s_2 is occupied, and this can only correspond to (s_2, p_1). Thus, $N(s_1, p_3)$ is $\sum_{i=1}^{2} N(s_1, p_i) = 2$. Similarly, for evaluating $N(s_1, p_4)$, we observe that p_4 will choose to occupy s_1 iff he finds *his* seat occupied, and only three people could have been the required occupant. Thus we will need to find the number of ways in which p_3 can find his seat occupied, and this will correspond to p_3 occupying s_4; the number of ways in which p_2 can find his seat occupied, and this will correspond to p_2 occupying s_4; and finally, the number of ways in which p_1 can find his seat occupied, and this will correspond to p_1 occupying s_4. The total number therefore amounts to $1 + 1 + 2 = 4$. In general, as the reader may have deduced by now, we have:

$$N(s_1, p_k) = \sum_{i=1}^{k-1} N(s_1, p_i).$$

This definition is recursive. In the following calculation, we repeatedly use the result:

$$1 + \sum_{i=0}^{k} 2^i = 2^{i+1}.$$

Observe that:

$$\begin{array}{lcc}
N(s_1, p_1) & = & 1 \\
N(s_1, p_2) & = 1 = & 2^0 \\
N(s_1, p_3) & = 1 + 2^0 = & 2^1 \\
N(s_1, p_4) & = 1 + 2^0 + 2^1 = & 2^2 \\
N(s_1, p_5) & = 1 + 2^0 + 2^1 + 2^2 = & 2^3 \\
 & \vdots & \\
N(s_1, p_k) & = 1 + \sum_{i=0}^{k-3} 2^i = & 2^{k-2} \\
 & \vdots & \\
N(s_1, p_{100}) & = 1 + \sum_{i=0}^{100-3} 2^i = 2^{100-2} = & 2^{98}
\end{array} \qquad (1)$$

Now all that remains to be seen is which combinations correspond to (s_1, p_{100}) and which correspond to (s_{100}, p_{100}), since we have already seen that every combination corresponds to one of these two situations. Observe that when (s_1, p_k) (for $k \neq 100$), then (s_i, p_i) for $i \in \{k+1, k+2, \ldots, 100\}$. Therefore, $N(s_1, p_i)$ $\forall i \in \{1, 2, \ldots, 99\}$ would give the total number of combinations such that (s_{100}, p_{100}) and $N(s_1, p_{100})$ corresponds to the case of (s_1, p_{100}). Actual calculation tells us that

$$\sum_{i=1}^{99} N(s_1, p_i) = 1 + 2^0 + 2^3 + \cdots + 2^{97} = 2^{98} \qquad (1a)$$

$$N(s_1, p_{100}) = 2^{98} \qquad (1b)$$

Thus there are 2^{98} valid combinations!