Hardness of r-DOMINATING SET on Graphs of Diameter $(r+1)$

D Lokshtanov[1], N Misra[2], G Philip[3], M S Ramanujan[4], and S Saurabh[1,4]

[1]*University of Bergen, Bergen, Norway*
[2]*Indian Institute of Science, Bengaluru, India*
[3]*Max-Planck-Institut für Informatik, Saarbrücken, Germany*
[4]*The Institute of Mathematical Sciences, Chennai, India*

IPEC 2013
Sophia Antipolis, France
September 06, 2013

Definitions

- The *distance* between vertices u, v in graph G
  - $d(u,v)$ = length (# edges) of a shortest path from u to v
- The *diameter* of graph G
  - $diam(G)$ = max distance between two vertices $u, v \in V(G)$
    - The length of a longest shortest-path
  - Examples:
    - A complete graph on ≥ 2 vertices has diameter 1
    - A path of length ℓ has diameter ℓ

Definitions

- A *dominating set* of graph G
  - A set $S \subseteq V(G)$; every $v \in V(G)$ is of distance at most 1 from some vertex in S
- The Dominating Set problem
  - Input: Graph G, integer k
  - Parameter: k
  - Question: Does G have a dominating set of size at most k?
- The Dominating Set problem is W[2]-hard
  - Downey and Fellows, 1995

Definitions

- For $r \geq 1$, an *r-dominating set* of graph G
  - A set $S \subseteq V(G)$; every $v \in V(G)$ is of distance at most r from some vertex in S
  - Generalization of dominating set
- The r-DOMINATING SET problem
  - Fixed integer $r \geq 1$
  - Input: Graph G, integer k
  - Parameter: k
  - Question: Does G have an r-dominating set of size at most k?
- The r-DOMINATING SET problem is W[2]-hard
  - For any fixed $r \geq 1$

This Work

Dominating Set and r-Dominating Set in graphs of bounded diameter

- What happens to these problems when the diameter of G is small?

This Work

DOMINATING SET and r-DOMINATING SET in graphs of bounded diameter

- What happens to these problems when the diameter of G is small?
- DOMINATING SET is trivial in graphs of diameter 1.

This Work

DOMINATING SET and r-DOMINATING SET in graphs of bounded diameter

- What happens to these problems when the diameter of G is small?
- DOMINATING SET is trivial in graphs of diameter 1.
  - What if the diameter is 2?

This Work

Dominating Set and r-Dominating Set in graphs of bounded diameter

- What happens to these problems when the diameter of G is small?
- Dominating Set is trivial in graphs of diameter 1.
  - What if the diameter is 2?
- r-Dominating Set is trivial in graphs of diameter r.
  - What if the diameter is $r + 1$?

Our Results

Both problems are hard ...

- DOMINATING SET is W[2]-hard in graphs of diameter 2.
- For any fixed integer $r \geq 1$, r-DOMINATING SET is W[2]-hard in graphs of diameter $r + 1$.

Brief Proof Outlines

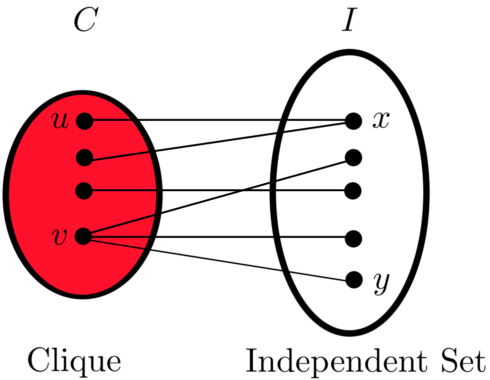
DOMINATING SET in graphs of diameter 2

W[2]-hardness

- Reduce from DOMINATING SET
- Make diameter $= 2$
  - A path of length ≤ 2 from any vertex to any other vertex
- How?
  - Reduce from DOMINATING SET on *split* graphs

DOMINATING SET in graphs of diameter 2

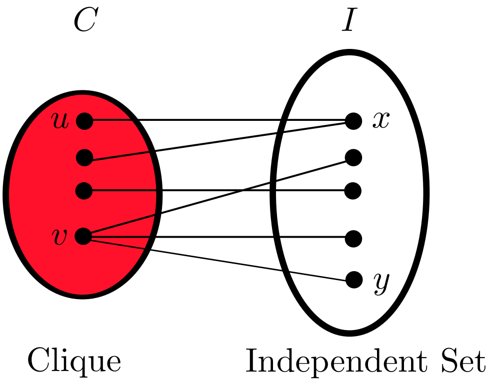
W[2]-hardness



- Split graph: $G = ((C \uplus I), E)$
  - C is a clique, I an independent set
  - Like a bipartite graph, but one part is a clique
- A connected split graph has diameter ≤ 3

Dominating Set in graphs of diameter 2

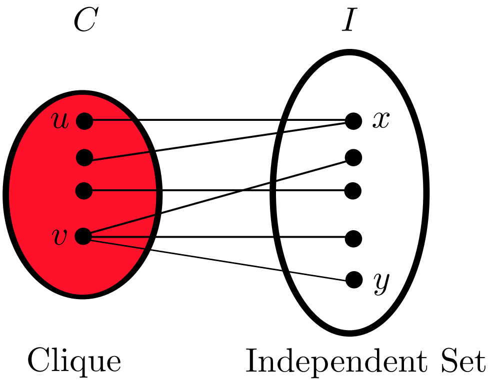
W[2]-hardness



- Dominating Set is W[2]-hard in connected split graphs
  - Raman and Saurabh, 2008
- A connected split graph has diameter ≤ 3

Dominating Set in graphs of diameter 2

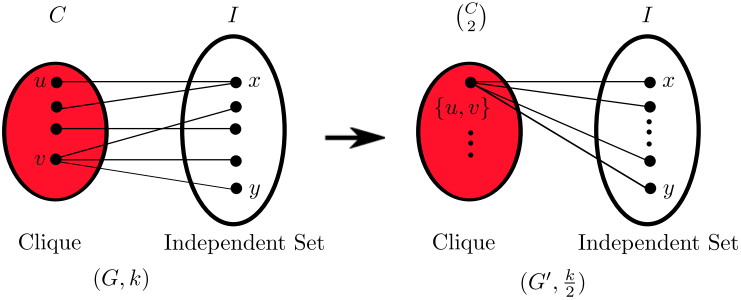
W[2]-hardness



- We want to add a 2-path between every pair $\{x, y\} \subseteq I$
  - Without ending up with a *really* small dominating set
- Can "push" a minimum dominating set S to the clique
  - May assume $S \subseteq C$

Reduction from Dominating Set in split graphs

Making diameter $= 2$



Theorem

Dominating Set *is W[2]-hard in split graphs of diameter* 2.

r-DOMINATING SET in graphs of diameter $r+1$

W[2]-hardness

- Reduce from DOMINATING SET in split graphs of diameter 2.
- Embed such a graph in a large "wireframe graph" which
  - Makes the diameter go from 2 to $r+1$
  - Takes a dominating set to an r-dominating set
    - Without blowing up or shrinking the solution size

r-DOMINATING SET in graphs of diameter $r+1$

The reduction

- Input: Instance (G, k) of DOMINATING SET
  - G is a split graph of diameter 2
- Output: Instance (G', k) of r-DOMINATING SET
  - G' has diameter $r+1$
  - $(G, k) \in$ DOMINATING SET $\iff$ $(G', k) \in r$-DOMINATING SET
- How?
  - Construct a wireframe graph W
    - W has diameter $r+1$
  - Embed G in W to get G'

The wireframe W

We want: diameter$(W) = r + 1$

- Input graph G; $V(G) = C \uplus I$
- Define: $\alpha = |C| + 4kr|I|$
  - $\alpha = \mathcal{O}(k|V(G)|)$

The wireframe W

We want: diameter$(W) = r + 1$

- Input graph G; $V(G) = C \uplus I$
- Define: $\alpha = |C| + 4kr|I|$
  - $\alpha = \mathcal{O}(k|V(G)|)$
- Vertex set of W:
  - All strings of length $r + 1$ over $[\alpha]$
  - $V(W) = \{1, 2, \ldots, \alpha\}^{(r+1)}$
- Examples of vertices:
  - $(1, 1, \ldots, 1)$
  - $(\alpha, \alpha, \ldots, \alpha)$
  - $(1, 2, \ldots, r + 1)$

The wireframe W

We want: $\text{diameter}(W) = r + 1$

- Input graph G; $V(G) = C \uplus I$
- Define: $\alpha = |C| + 4kr|I|$
  - $\alpha = \mathcal{O}(k|V(G)|)$
- Edge set of W:
  - All pairs of vertices which differ at *exactly* one position
  - $E(W) = \{(u, v) \in V(W) \times V(W) \mid HamDist(u, v) = 1\}$
    - $HamDist(u, v) =$ Hamming distance between u and v
- Examples:
  - Edge: $((1, 1, \ldots, 1), (2, 1, 1, \ldots, 1)$
  - Edge: $((1, 1, \ldots, 1), (1, \alpha, 1, \ldots, 1)$
  - Non-edge: $((1, 1, \ldots, 1), (2, 2, \ldots, 1)$

The wireframe W

We want: diameter(W) $= r + 1$

- Input graph G; $V(G) = C \uplus I$
- Define: $\alpha = |C| + 4kr|I|$
  - $\alpha = \mathcal{O}(k|V(G)|)$

- Vertex set of W:
  - All strings of length $r + 1$ over $[\alpha]$
  - $V(W) = \{1, 2, \ldots, \alpha\}^{(r+1)}$
- Edge set of W:
  - All pairs of strings which differ at *exactly* one position
  - $E(W) = \{(u, v) \in V(W) \times V(W) \mid \mathit{HamDist}(u, v) = 1\}$
    - $\mathit{HamDist}(u, v) =$ Hamming distance between u and v
- The wireframe W is a "hypercube"

Some simple properties of the wireframe W

- Distance between two vertices u, v of $W = HamDist(u, v)$
- Diameter of $W = r + 1$
  - Two vertices can differ at at most $r + 1$ positions
  - $(1, 1, \ldots, 1), (2, 2, \ldots, 2)$

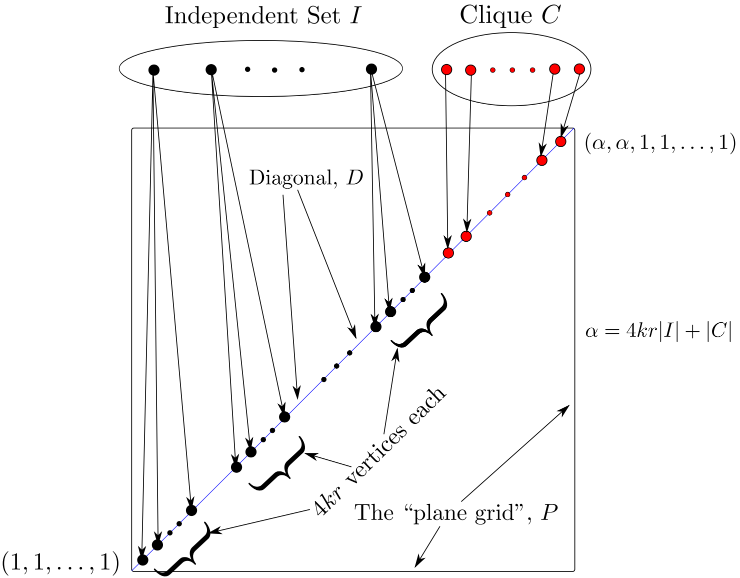
Embedding G in W to get G'

G is a split graph of diameter 2

- Define $P = \{(a, b, 1, 1, \ldots, 1)\} \subseteq V(W)$
  - All vertices whose last $r - 1$ coordinates are 1
  - P is the "plane grid"
- Let $D = \{(a, a, 1, 1, \ldots, 1)\} \subseteq P$
  - D is a diagonal of the plane grid
- We map $V(G) = C \uplus I$ to vertices in the plane diagonal D

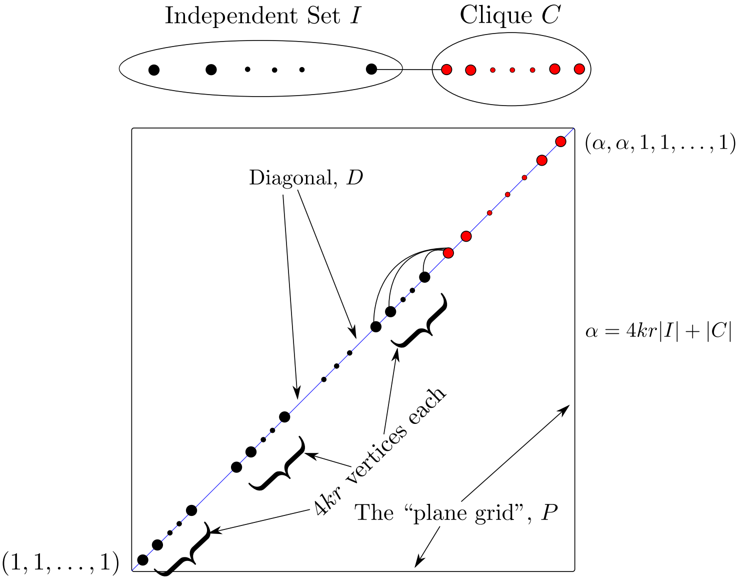
Embedding G in W to get G'

Mapping $V(G) = I \uplus C$ onto the plane diagonal



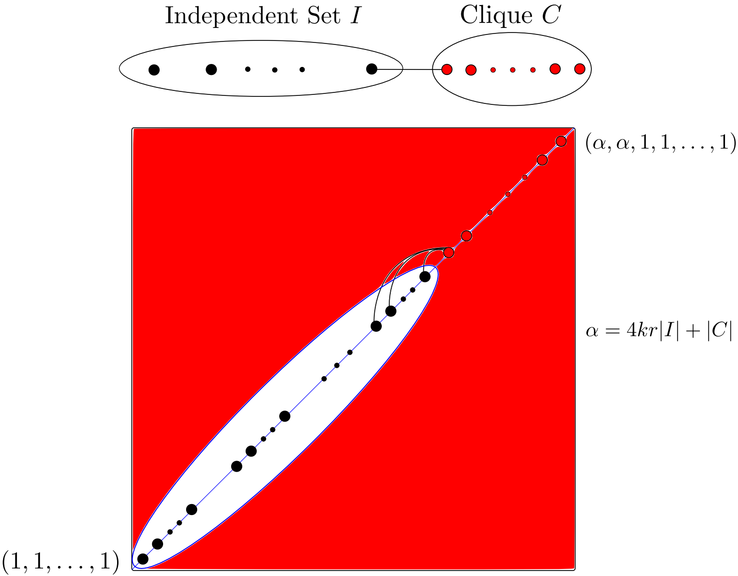
Embedding G in W to get G'

Mapping $E(G)$ onto the plane diagonal



Embedding G in W to get G'

Mapping $E(G)$ onto the plane diagonal

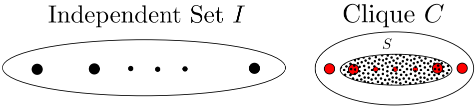


G' has diameter $r + 1$

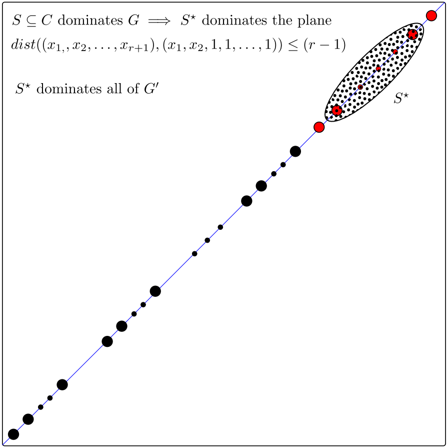
- We only added edges to the plane
- Shortest paths which are *far from the plane* are not affected
  - e.g: Shortest path from $(2, 2, \ldots, 2)$ to $(3, 3, \ldots, 3)$

Correctness of the reduction

Forward direction



- The copy $S^\star$ of any dominating set $S \subseteq C$ dominates all of the plane
- Vertex x not in the plane $\implies$ Exists vertex $x^\star$ in the plane with $distance(x, x^\star) \leq (r-1)$
  - Just "project" x down to the plane to get $x^\star$
- $S^\star$ r-dominates all of G'



Correctness of the reduction

Reverse direction

- Let S'; $|S'| \leq k$ be an r-dominating set of G'
- Each "independent set" vertex v in G has a copy $v^\star$ in $V(G')$:
  - There must exist a vertex $x \in S'$
    - x is adjacent to $v^\star$, and
    - x is in the plane
- (Simplistic) Reason:
  - v has too many copies for S' to dominate from "far away"

Correctness of the reduction

Reverse direction

- Let S'; $|S'| \leq k$ be an r-dominating set of G'
- Each "independent set" vertex v in G has a copy $v^\star$ in $V(G')$:
  - There must exist a vertex $x \in S'$
    - x is adjacent to $v^\star$, and
    - x is in the plane
- Let $Y = \bigcup_{v \in I} v^\star$
  - Y is a set of vertices in the plane
  - $|Y| \leq k$
  - Y dominates (at least) one copy of each independent set vertex
- Y can be pulled back into the "clique image" to get S
  - Using the split graph structure of the plane grid
  - S dominates all of G

Correctness of the reduction

Reverse direction

- Let $S'; |S'| \leq k$ be an r-dominating set of G'
- Each "independent set" vertex v in G has a copy $v^\star$ in $V(G')$:
  - There must exist a vertex $x \in S'$
    - x is adjacent to $v^\star$, and
    - x is in the plane
- Let $Y = \bigcup_{v \in I} v^\star$
  - Y is a set of vertices in the plane
  - $|Y| \leq k$
  - Y dominates (at least) one copy of each independent set vertex
- Y can be pulled back into the "clique image" to get S
  - Using the split graph structure of the plane grid
  - S dominates all of G

Theorem

r-Dominating Set *is W[2]-hard in graphs of diameter* $(r+1)$.

Conclusion

- r-DOMINATING SET is W[2]-hard on graphs of diameter $r + 1$, ...
  - ...though it is trivial to solve on graphs of diameter at most r
- The same reduction shows the hardness for the *connected* variant
- How do other W-hard problems behave on graphs of bounded diameter?