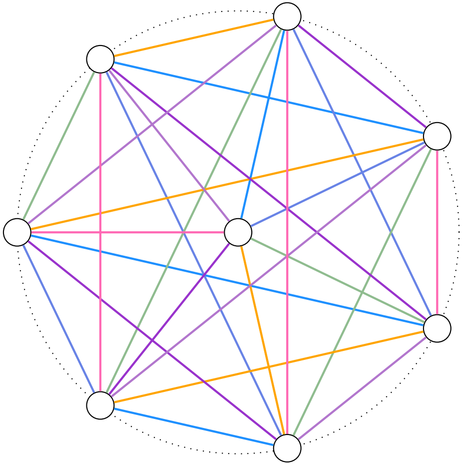
EKR for Matchings



$\mathcal{M}_n^r$: the family of all matchings of size r in K_{2n}.
(Note that $r \leqslant n$.)

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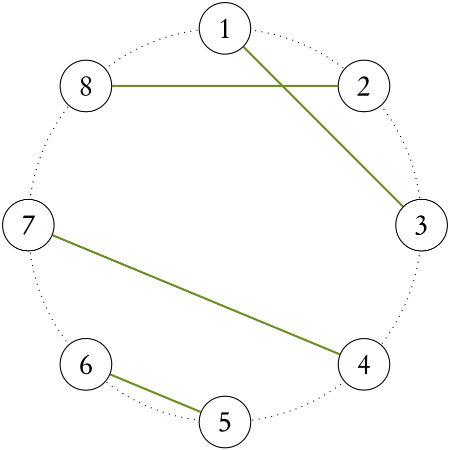

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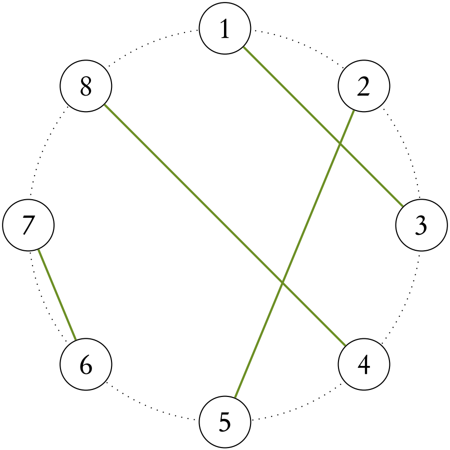

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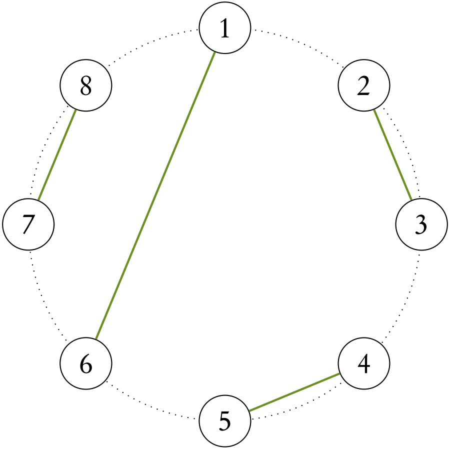

$\mathcal{M}_n^r(e)$: all matchings of K_{2n}, on r edges, containing e.
(This is the star centered at e.)

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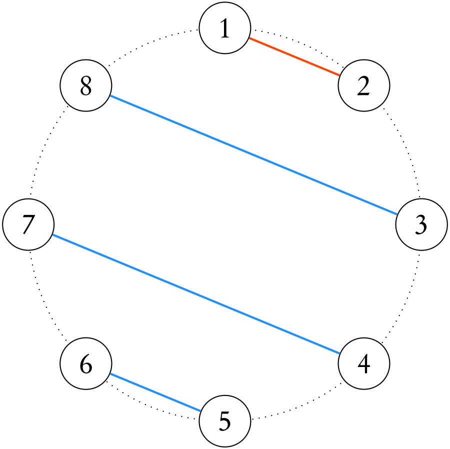

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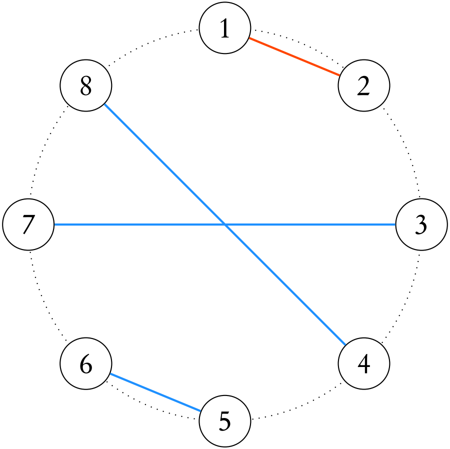

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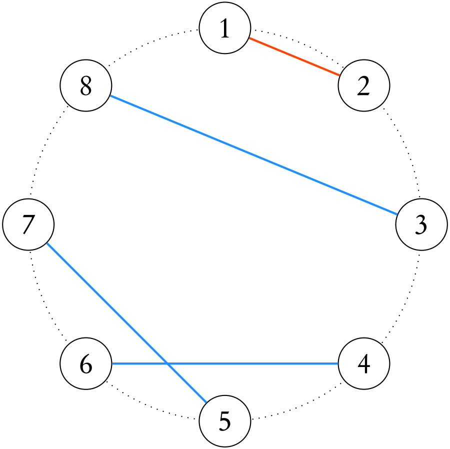

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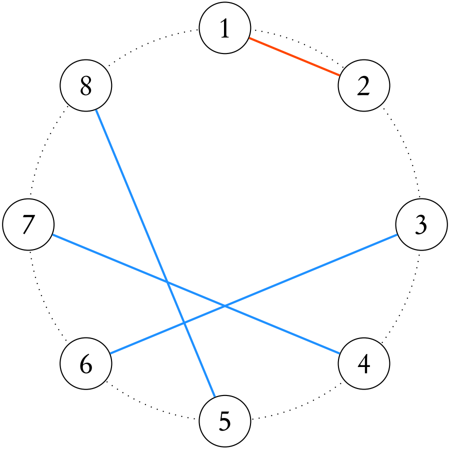

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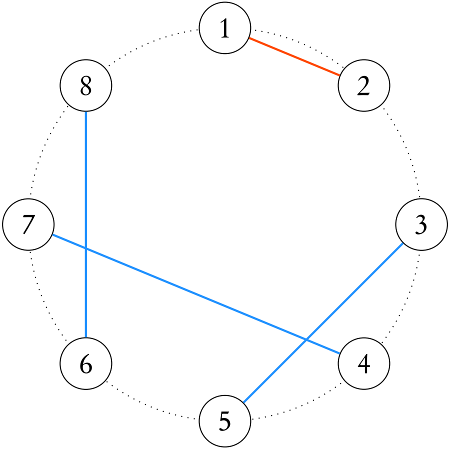

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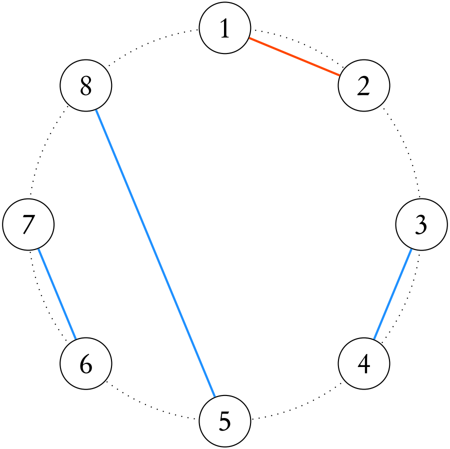

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EKR for Set Systems

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Proof by Picture

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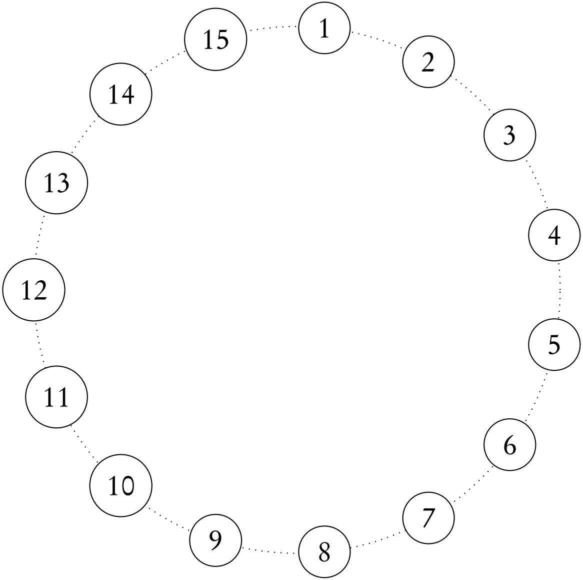
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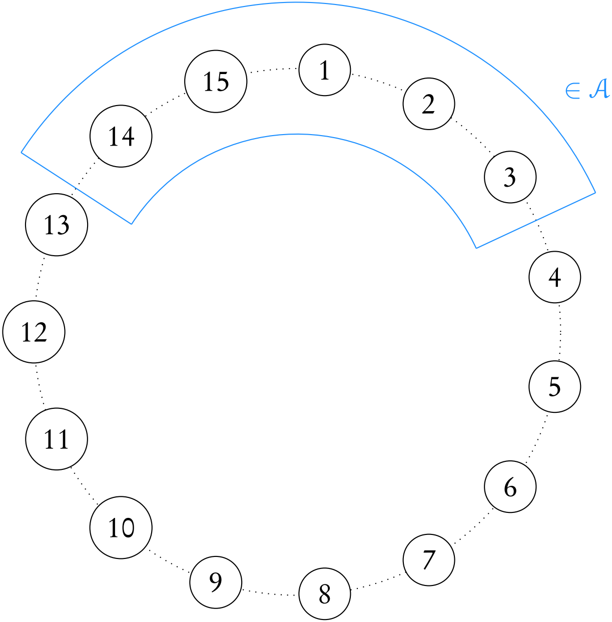

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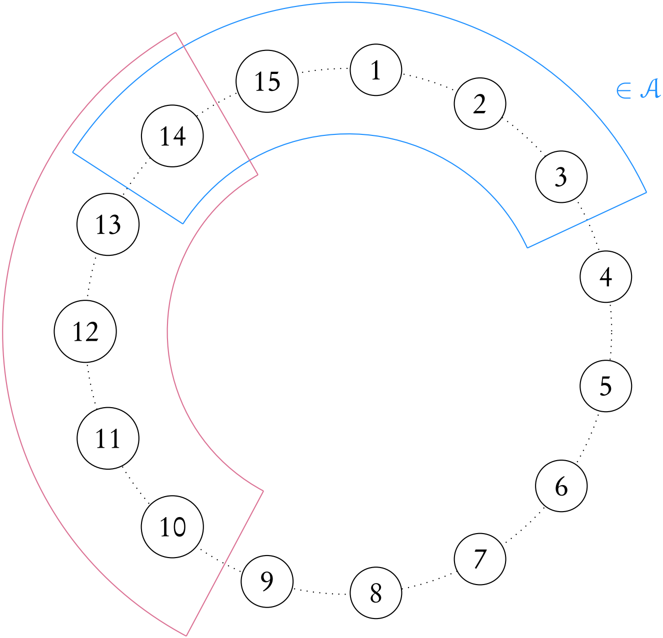
$\in \mathcal{A}$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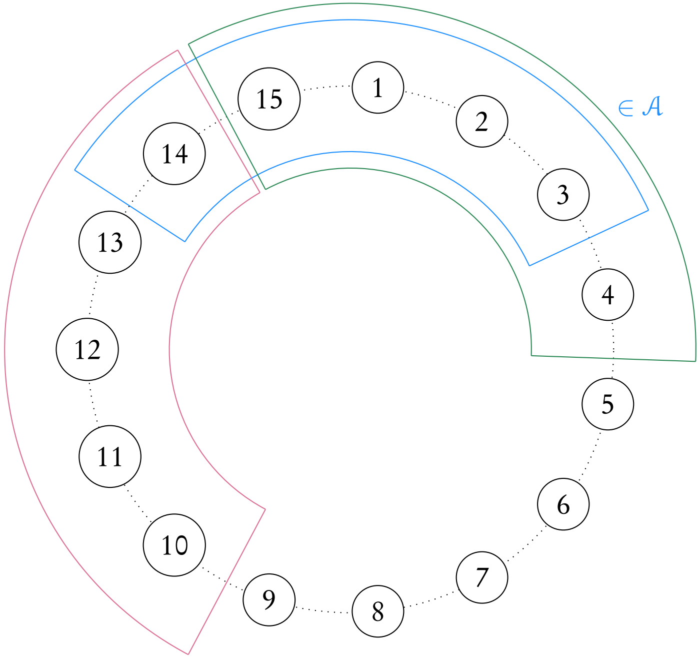
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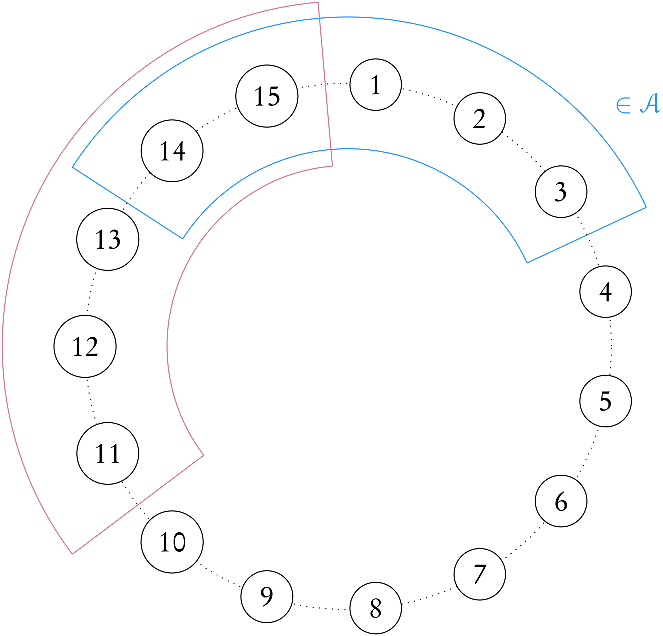

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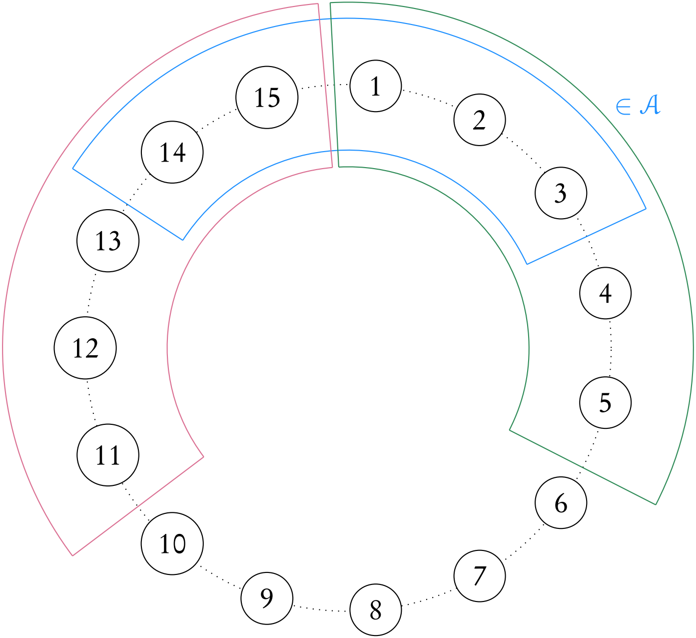
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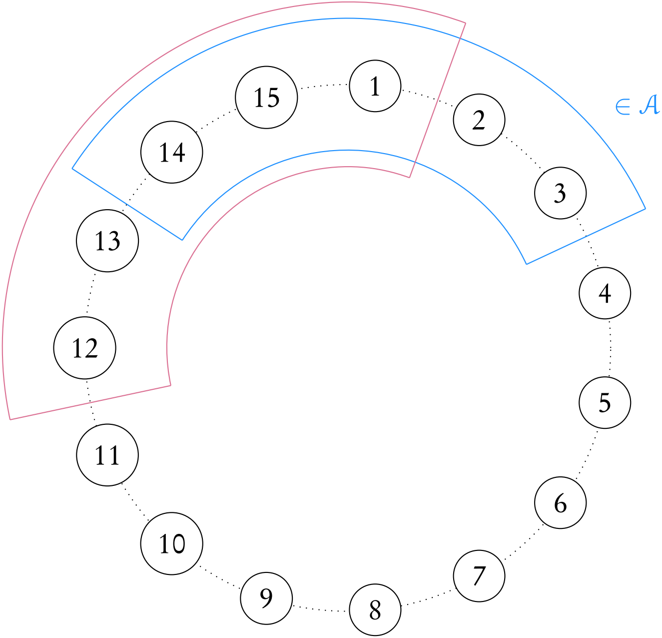
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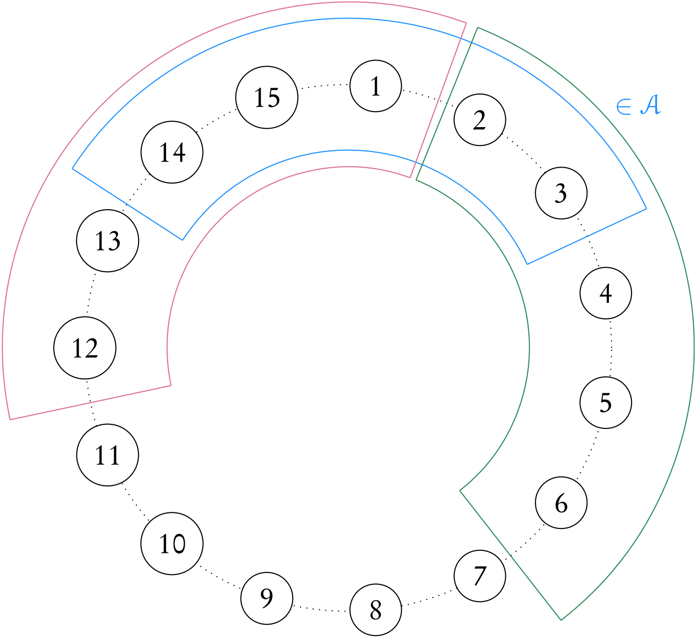

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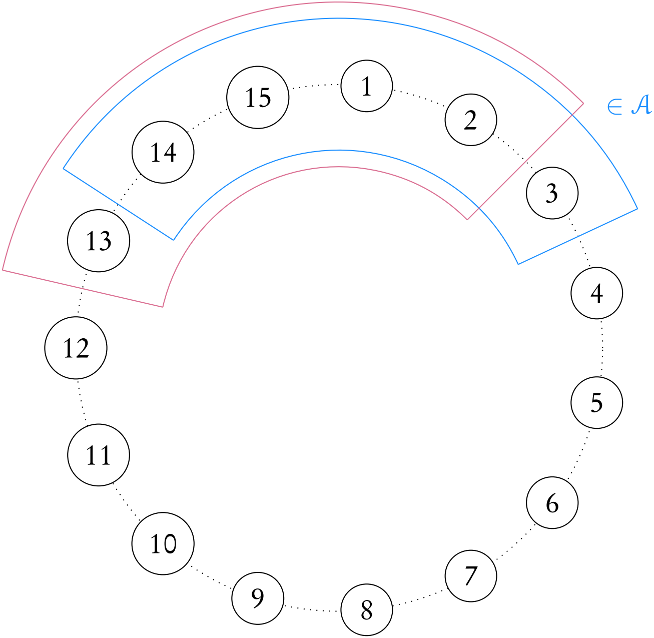

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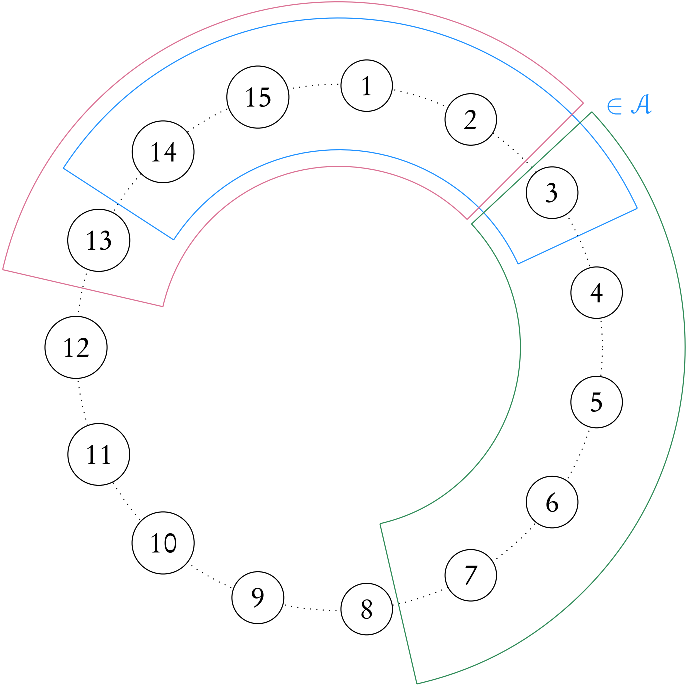

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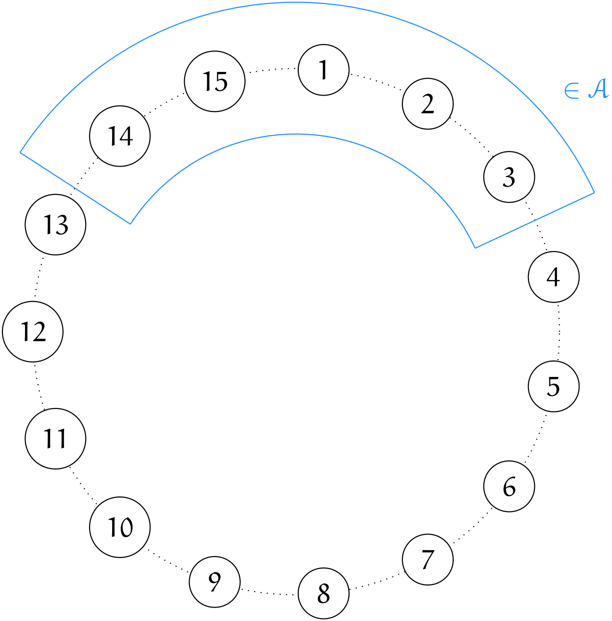
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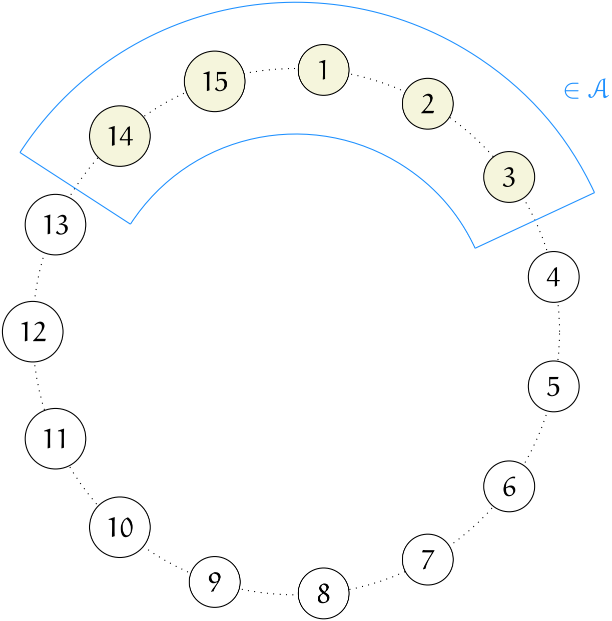
$\in \mathcal{A}$



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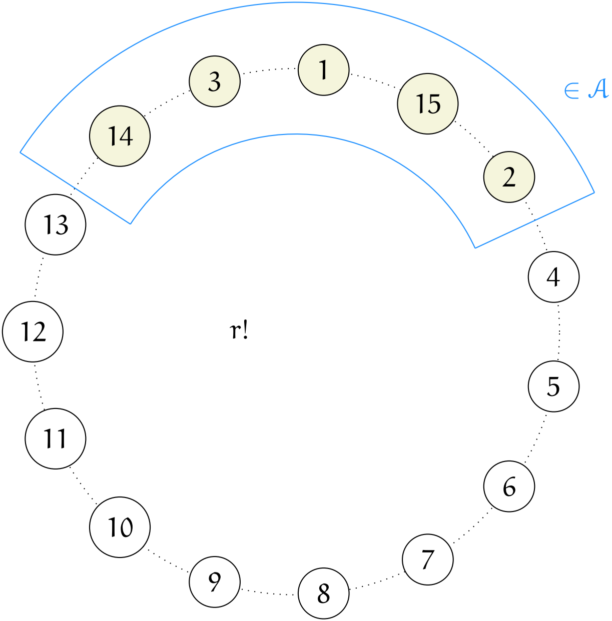
$\in \mathcal{A}$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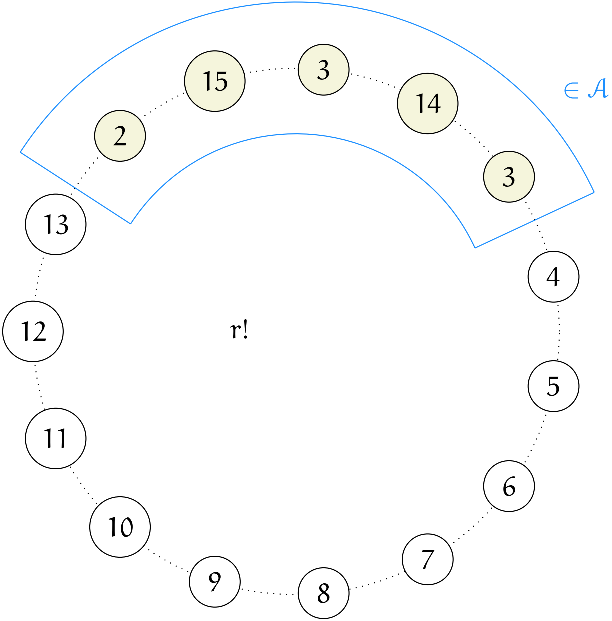
∈ 𝒜
r!



∈ $\mathcal{A}$
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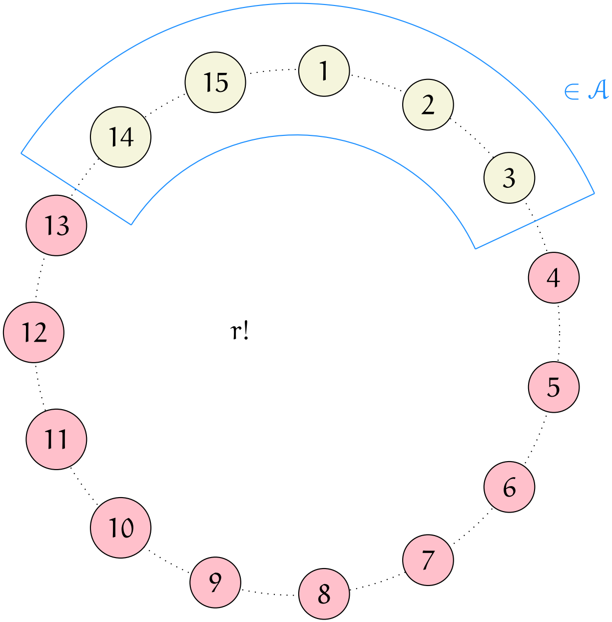

∈ $\mathcal{A}$
r!
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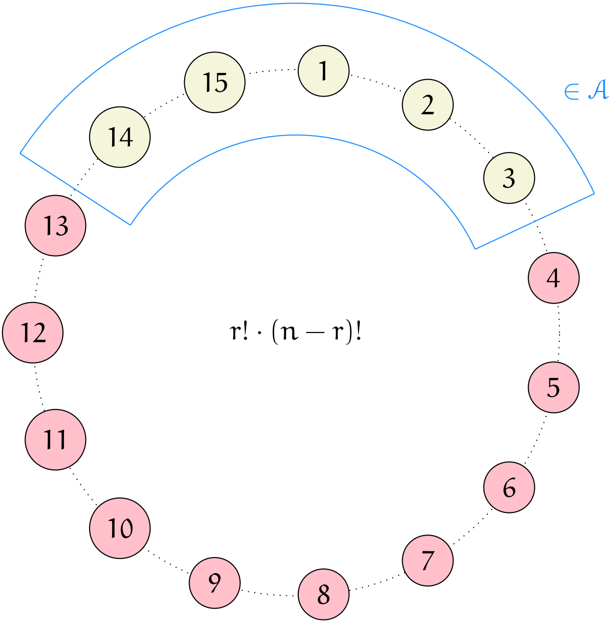

$\in \mathcal{A}$
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$r! \cdot (n-r)!$

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$$|\mathcal{A}| \leqslant \frac{(n-1)!r}{r!(n-r)!}$$

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$$|\mathcal{A}| \cdot r! \cdot (n-r)! \leqslant (n-1)!r$$

$$|\mathcal{A}| \leqslant \frac{(n-1)!r}{r!(n-r)!} = \binom{n-1}{r-1}.$$

Setting Up Katona-Like Local Environments

Our universe is now the set of all edges in K_{2n}.

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A direct approach only leads to a weak bound...

...and informally, the bounds are loose because
an interval of an arbitrary permutation of $E(K_{2n})$
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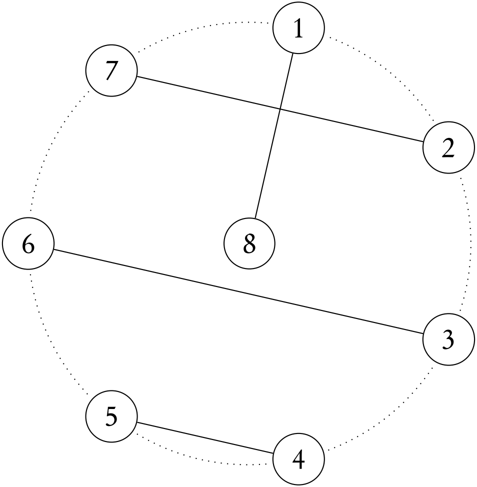
Our first goal, therefore, is to come up with a more suitable selection of cyclic permutations.

Baranyai Partitions

A Decomposition of the Edges of K_{2n}
into $(2n - 1)$ perfect matchings.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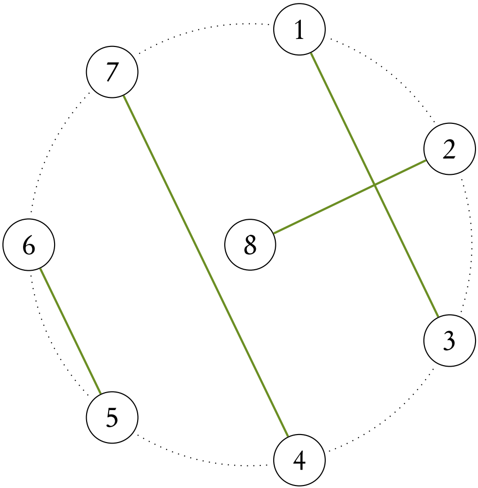
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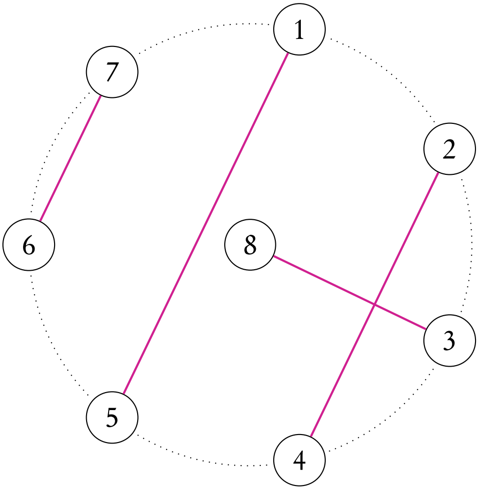
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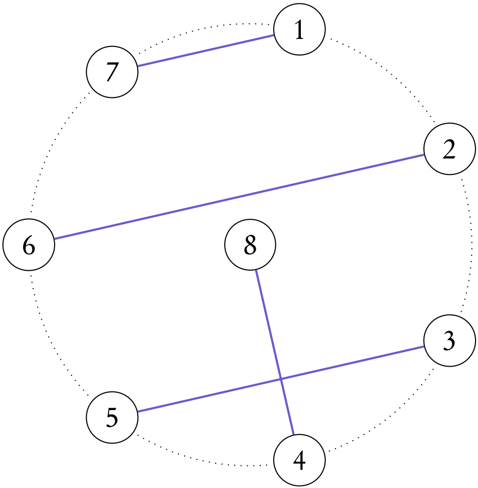
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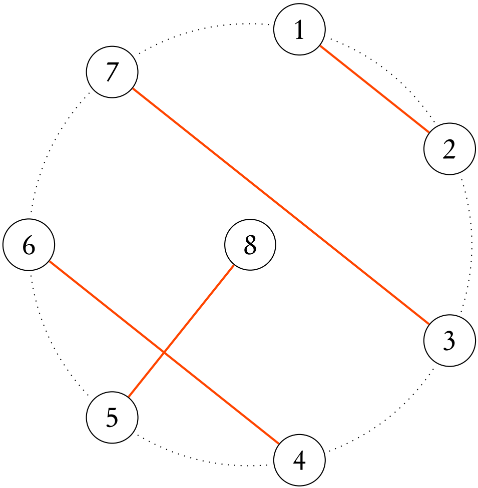
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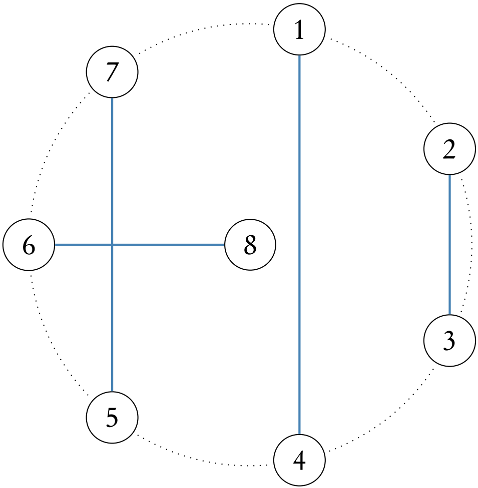
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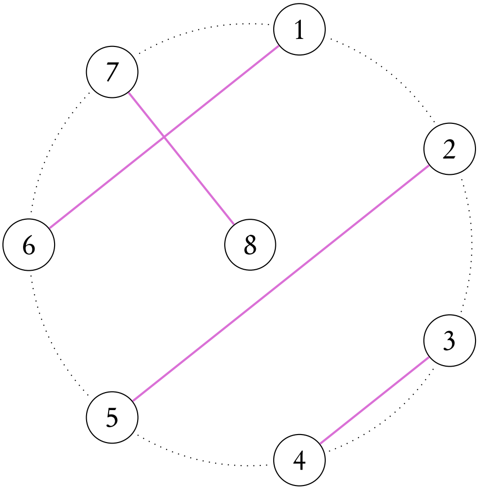
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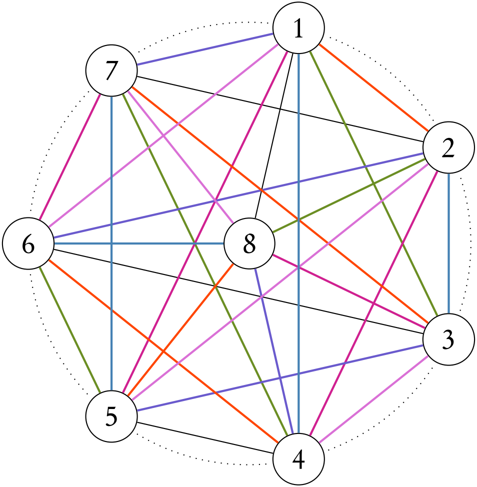
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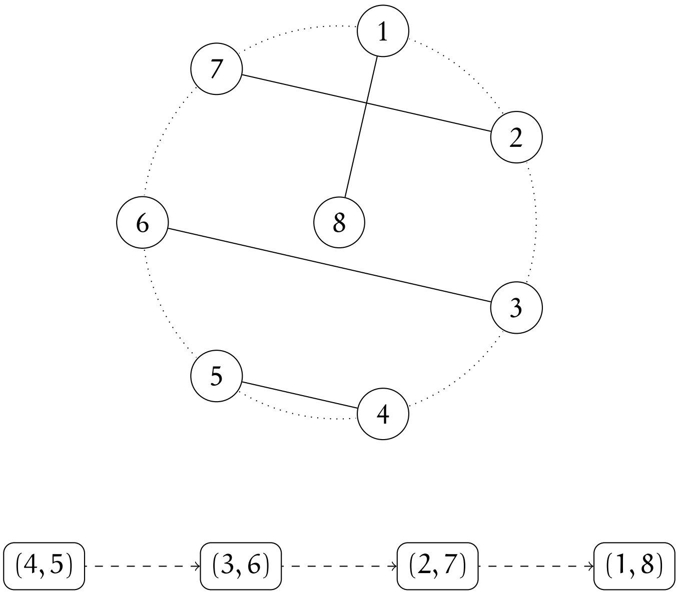
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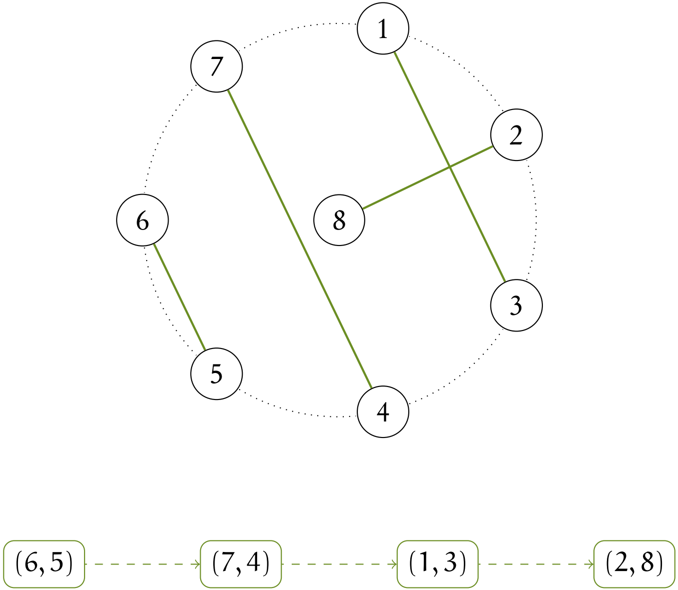
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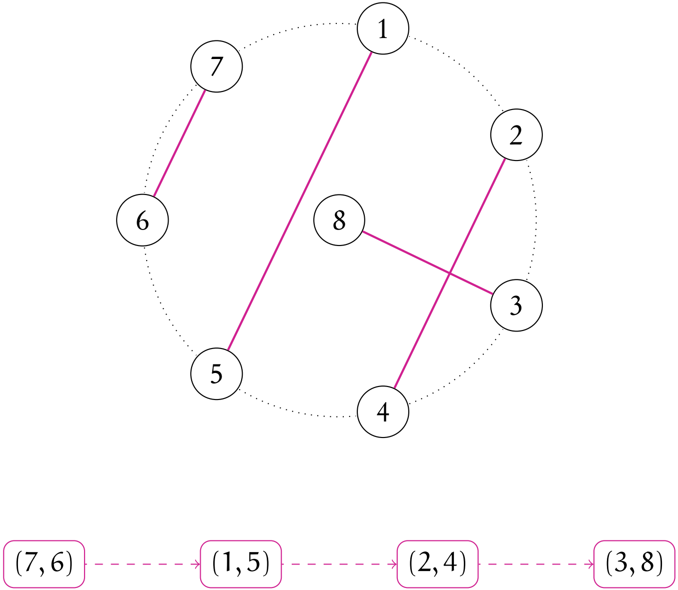
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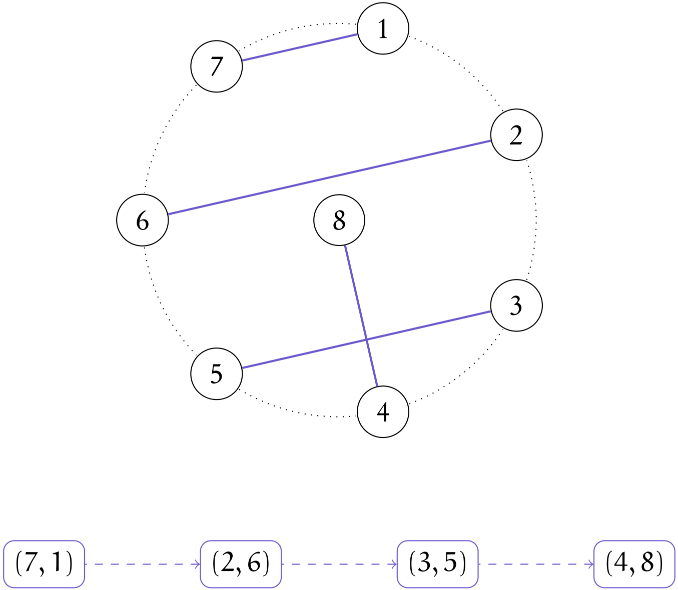
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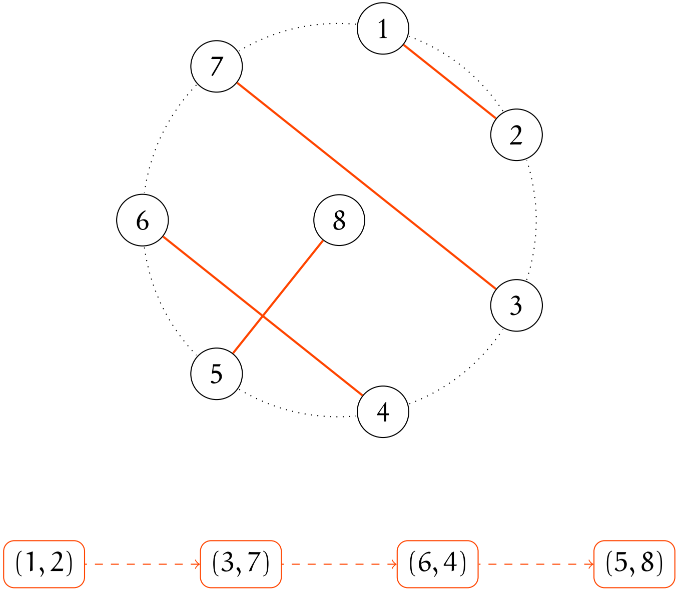
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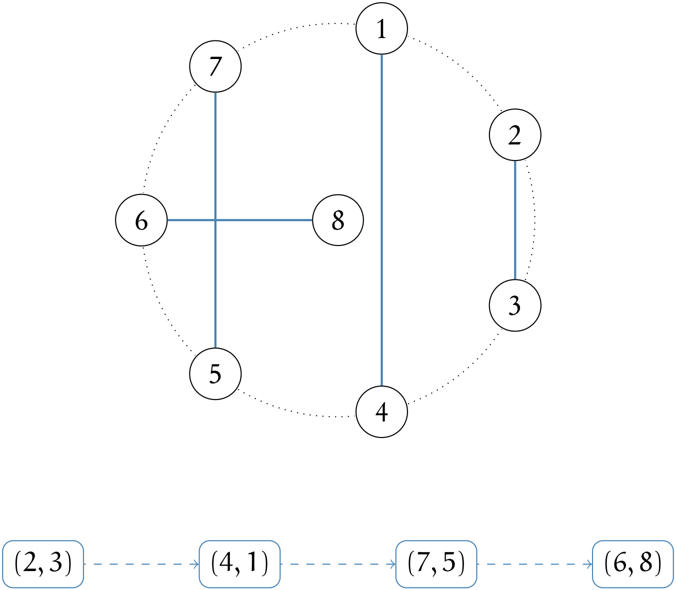
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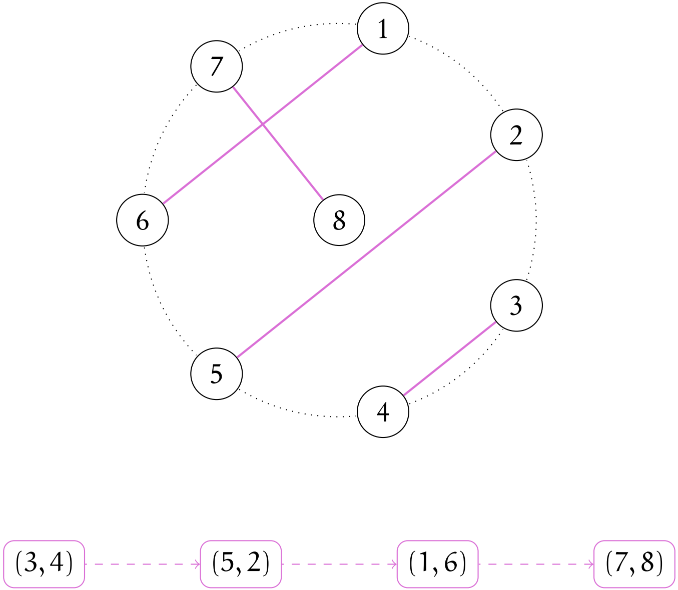
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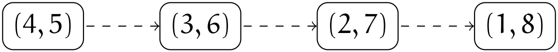
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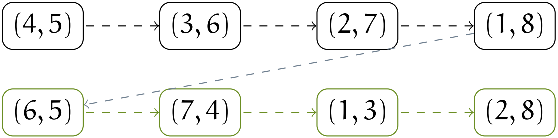
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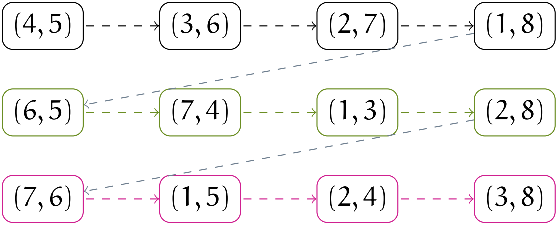

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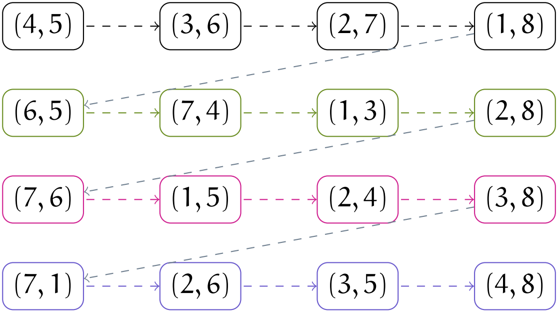
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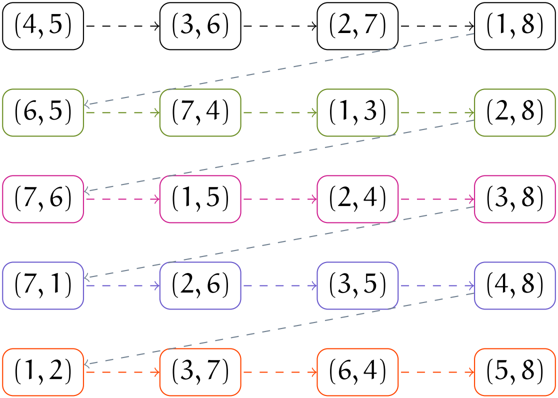
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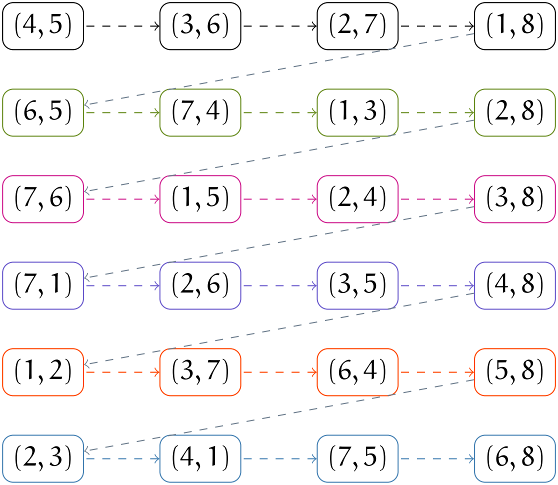
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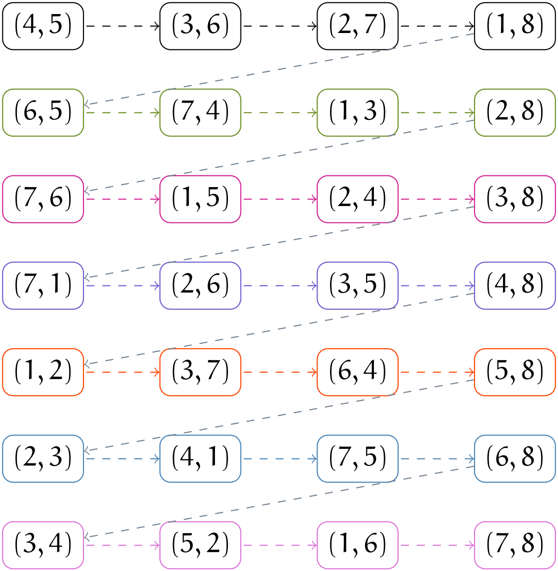
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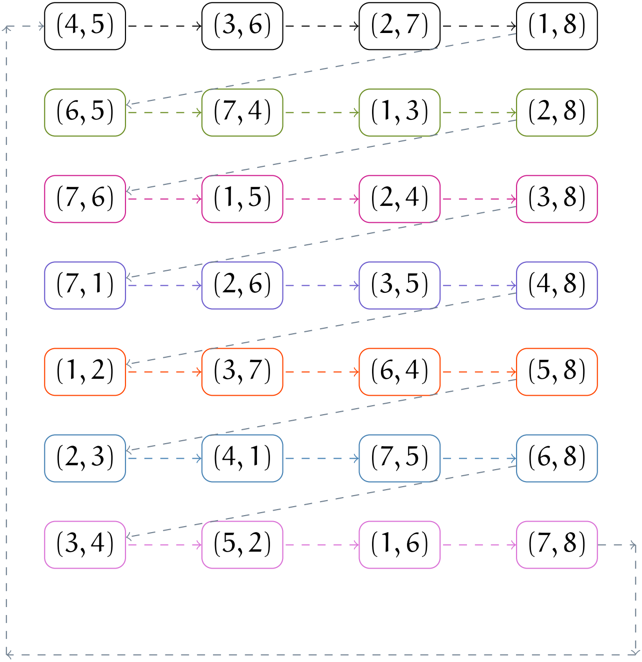
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We can generate other cyclic permutations of $E(K_{2n})$ using this method.

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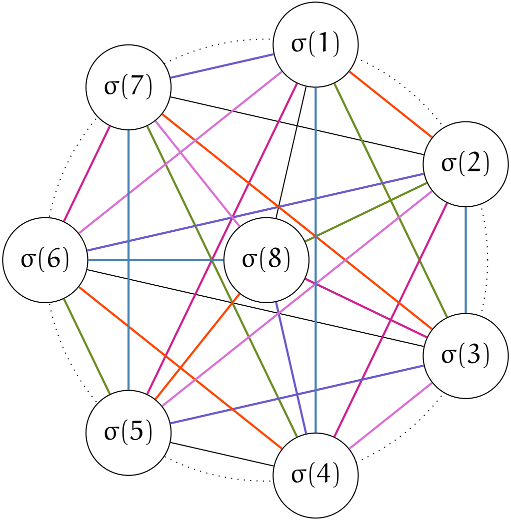
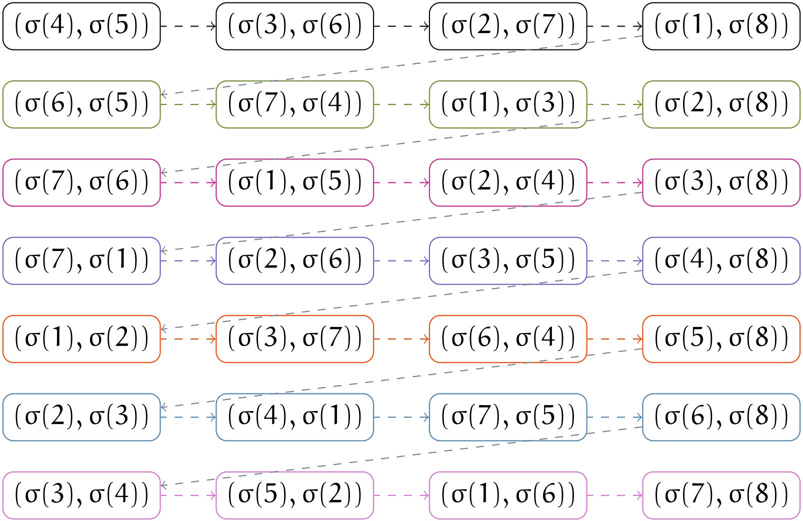
Let σ be a permutation of $[2n]$.
Start with the following Baranyai Partition...
(illustration for $n = 4$):



σ(1)
σ(2)
σ(3)
σ(4)
σ(5)
σ(6)
σ(7)
σ(8)

...and generate the following cyclic order, just as before:


(σ(4), σ(5))
(σ(3), σ(6))
(σ(2), σ(7))
(σ(1), σ(8))
(σ(6), σ(5))
(σ(7), σ(4))
(σ(1), σ(3))
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(σ(2), σ(6))
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The cyclic orders that we have just generated will serve as wireframes on which we can project elements of $\mathcal{A}$ as intervals, *a la* Katona's proof for the classic version of the theorem.

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Preliminary Observation.
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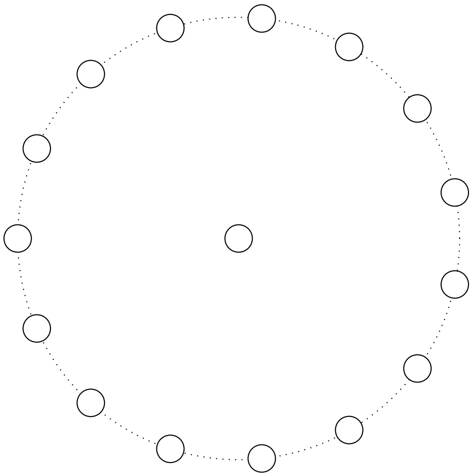

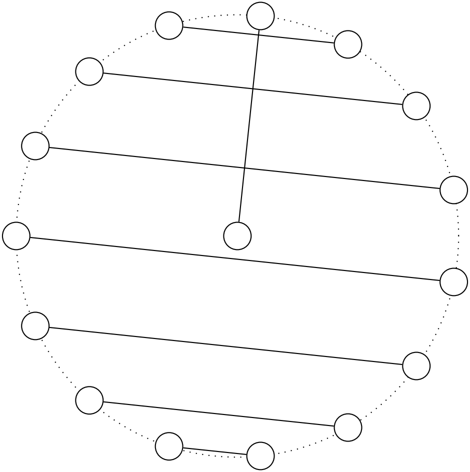

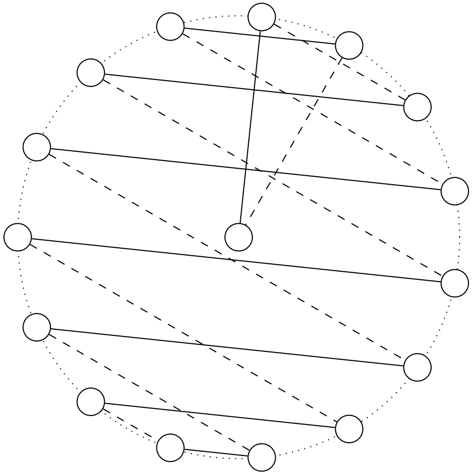

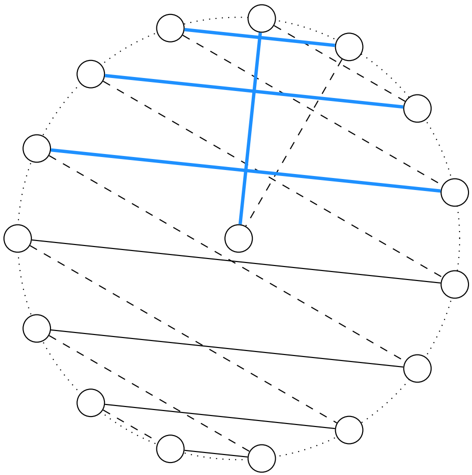

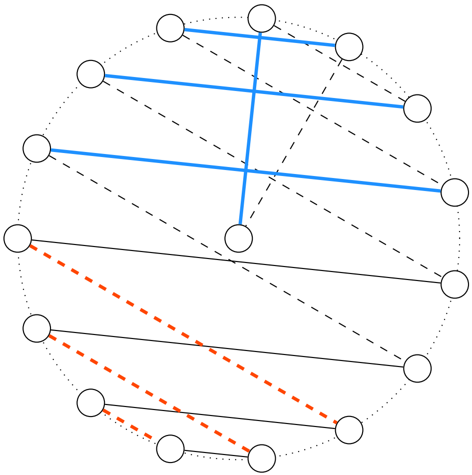

Proving the EKR bound

Let $\mathcal{A}$ be an intersecting family of r-matchings of K_{2n}, where $r < n$.

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Let σ be a permutation of $[2n]$ - consider the cyclic permutations of $E(K_{2n})$ that we generated based on σ - let's call this χ_σ.

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$\mathcal{A}_\sigma$: those sets of $\mathcal{A}$ that happen to be intervals on this circular arrangement.

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How big can $\mathcal{A}_\sigma$ be, given that $\mathcal{A}$ is intersecting?

Proving the EKR bound

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$\mathcal{A}_\sigma$: those sets of $\mathcal{A}$ that happen to be intervals on this circular arrangement.

$|\mathcal{A}_\sigma| \leqslant r$, for the same reasons as before.

Proving the EKR bound (contd.)

As before, consider the set of pairs (M, σ), where:

- M is a r-matching of K_{2n},

Proving the EKR bound (contd.)

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- M is a r-matching of K_{2n},
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Proving the EKR bound (contd.)

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- M belongs to $\mathcal{A}$,
- σ is a permutation of $[2n]$,
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Proving the EKR bound (contd.)

As before, consider the set of pairs (M, σ), where:

- M is a r-matching of K_{2n},
- M belongs to $\mathcal{A}$,
- σ is a permutation of $[2n]$,
- and M occurs as an interval in χ_σ.

Clearly,

$$\#(M, \sigma) \leqslant r \cdot (2n)!$$

Now, let us investigate the following question:

In how many cyclic orders χ_σ can a given matching M occur as an interval?

To address the question, let us recall the cyclic orders χ_σ.

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It is useful to think of χ_σ as being composed of $(2n-1)$ chunks, as they arise from the $(2n-1)$ parts of the Baranyai partitions.

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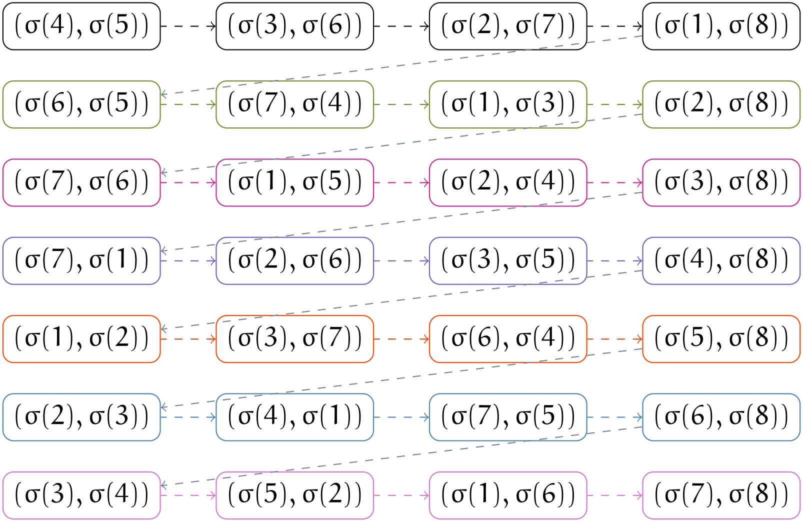
In the picture that follows, each row is a "chunk".


(σ(4), σ(5))
(σ(3), σ(6))
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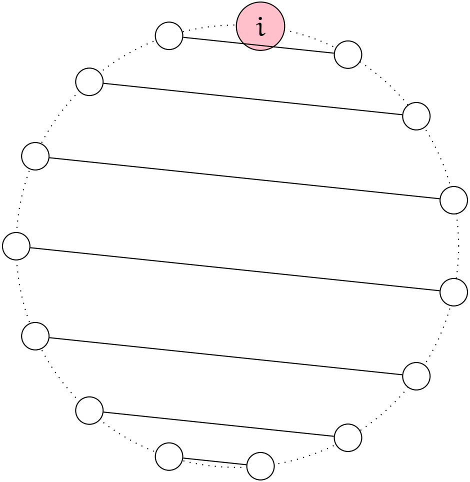
(σ(2), σ(3))
(σ(4), σ(1))
(σ(7), σ(5))
(σ(6), σ(8))
(σ(3), σ(4))
(σ(5), σ(2))
(σ(1), σ(6))
(σ(7), σ(8))

Task 1.

Enumerate all σ where
M belongs to χ_σ as an interval,
and M lies entirely inside one of the chunks of χ_σ.

If M has r edges, then we have $r!$ ways to order the edges of M.

For a fixed ordering p of the edges of M, let us devise a σ such that χ_σ will contain M as an interval in its i^{th} chunk, with the edges of M appearing in the order prescribed by p.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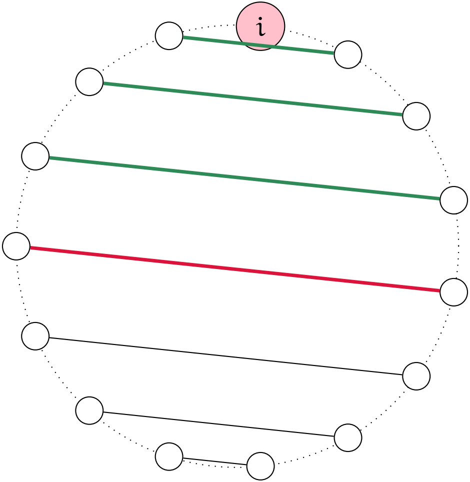
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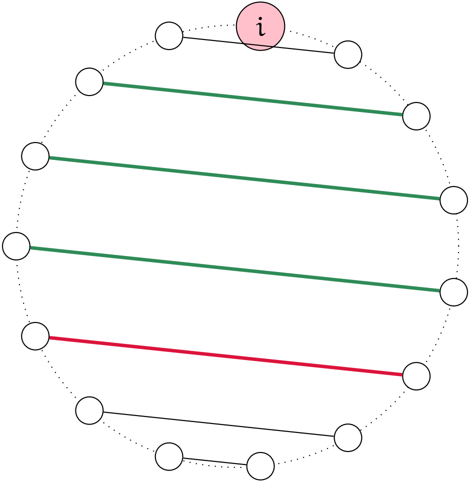
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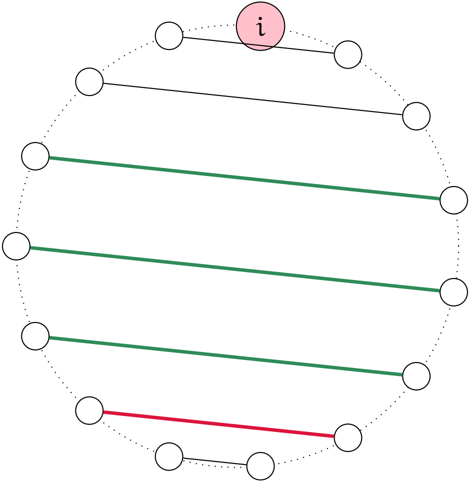
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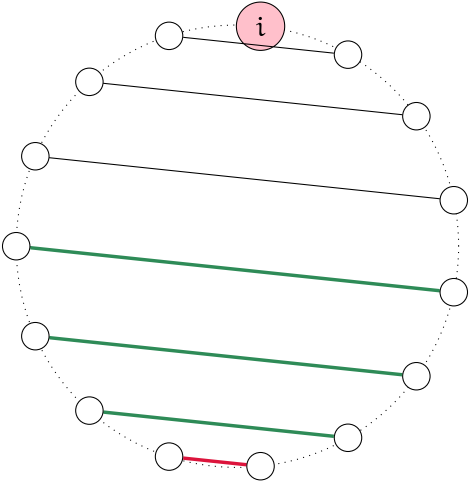
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We have $(n - r)$ choices to begin the placement of the edges of M.

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The ordering of the vertices that are not incident to M are immaterial, and there are $(2n - 2r)!$ such orderings.

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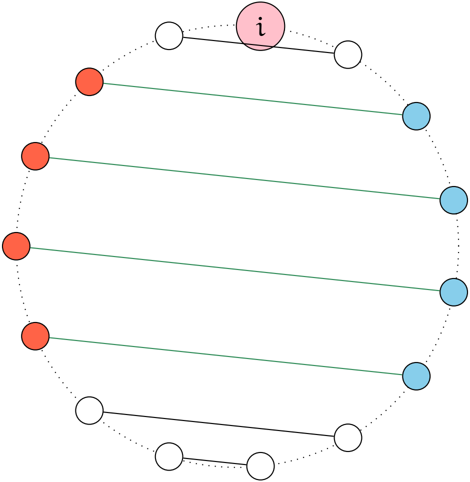
The ordering of the vertices that are not incident to M are immaterial, and there are $(2n - 2r)!$ such orderings.

Finally, for a fixed realization of M respecting the order p, we may still swap the endpoints of M to get a different permutation with the same realization.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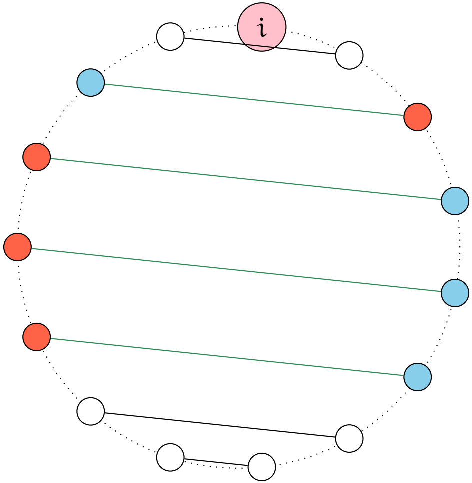
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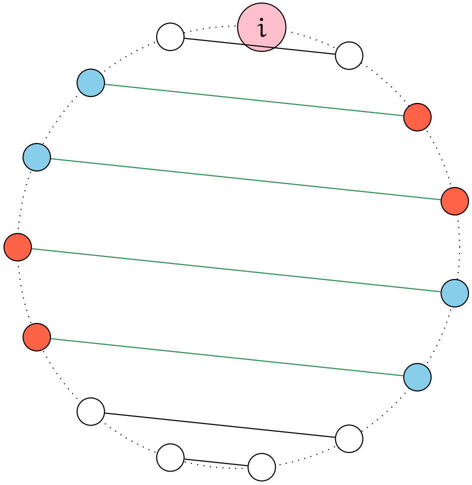
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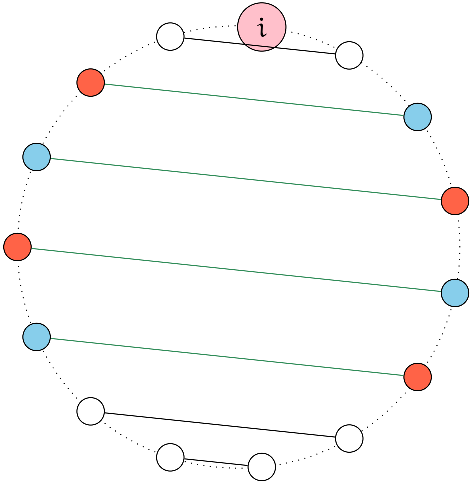
i

Note that there are 2^r choices for swapping the endpoints of the edges of M.

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Putting everything together, we have....

These many ways in which
the chunk of σ in which we realize M
can be chosen:

$(2n - 1)$

These many ways in which
the ordering of the edges of M
can be chosen:

$$(2n-1)\cdot r!$$

These many ways in which
the starting point of M
can be chosen:

$$(2n-1) \cdot r! \cdot (n-r)$$

These many ways in which
the ordering the vertices not incident to M
can be chosen:

$$(2n-1) \cdot r! \cdot (n-r) \cdot (2n-2r)!$$

These many ways in which
"swapped" vertices within edges of M
can be chosen:

$$(2n-1)\cdot r!\cdot (n-r)\cdot (2n-2r)!\cdot 2^r$$

These many ways in which
the permutation σ
can be chosen:

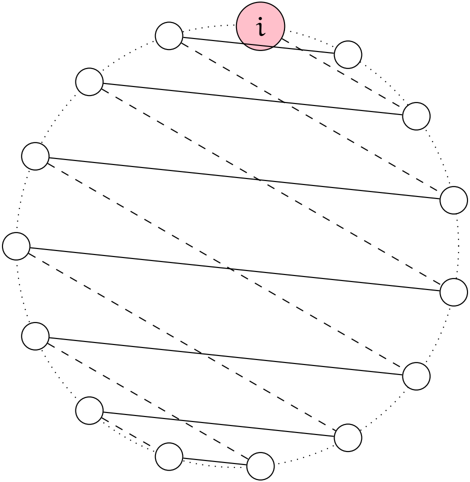
$$(2n-1)\cdot r!\cdot(n-r)\cdot(2n-2r)!\cdot 2^r$$

Task 2.

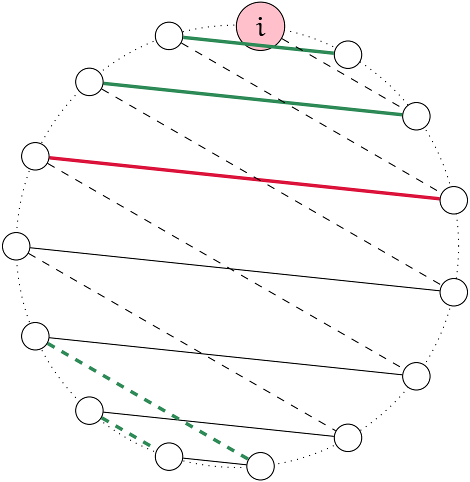
Enumerate all σ where
M belongs to χ_σ as an interval,
and M “splits across” two chunks of χ_σ.

If M has r edges, then we have $r!$ ways to order the edges of M.

For a fixed ordering p of the edges of M, let us devise a σ such that χ_σ will contain M as an interval starting in its i^{th} chunk, with the edges of M appearing in the order prescribed by p, and spilling over to the $(i+1)^{th}$ chunk.



(Crossover edge not depicted for clarity.)



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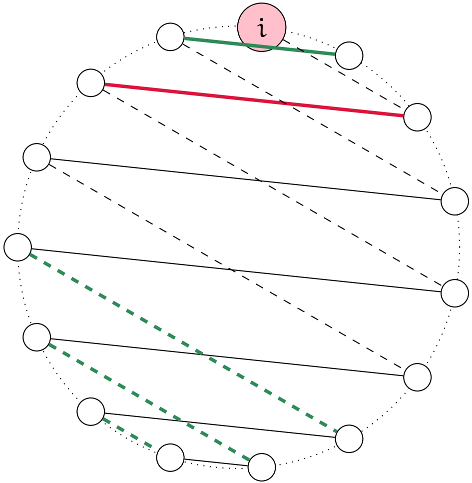

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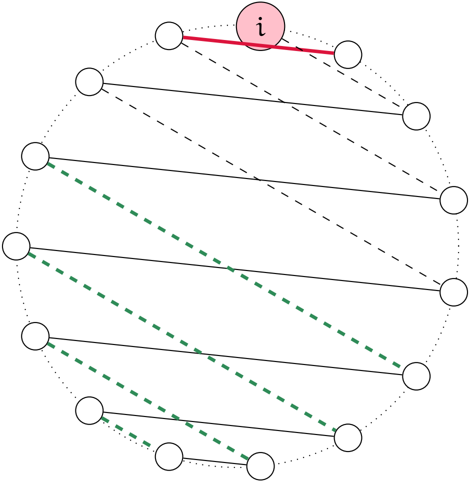

(Crossover edge not depicted for clarity.)

This time, we have r choices to begin the placement of the edges of M.

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And again, for a fixed realization of M respecting the order p, we may still swap (in 2^r ways) the endpoints of M to get a different permutation with the same realization.

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These many ways in which
the ordering the vertices not incident to M
can be chosen:

$$(2n-1) \cdot r! \cdot r \cdot (2n-2r)!$$

These many ways in which
“swapped” vertices within edges of M
can be chosen:

$$(2n-1) \cdot r! \cdot r \cdot (2n-2r)! \cdot 2^r$$

These many ways in which
the permutation σ
can be chosen:

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We are now in a position to tackle this:

In how many cyclic orders χ_σ can a given matching M occur as an interval?

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$$(2n-1)\cdot r!\cdot r\cdot(2n-2r)!\cdot 2^r +$$

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In how many cyclic orders χ_σ can a given matching M occur as an interval?

$$(2n-1)\cdot r!\cdot r\cdot(2n-2r)!\cdot 2^r +$$

$$(2n-1)\cdot r!\cdot(n-r)\cdot(2n-2r)!\cdot 2^r$$

$$= (2n-1)\cdot r!\cdot n\cdot(2n-2r)!\cdot 2^r$$

Finally, we have:

$$\#(M, \sigma) \leqslant (2n)!r$$

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$$|\mathcal{A}| \cdot (2n-1)r!n(2n-2r)!2^r = \#(M,\sigma) \leqslant (2n)!r$$

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$$|\mathcal{A}| \leqslant \frac{(2n)!r}{(2n-1)r!n(2n-2r)!2^r}$$

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$$|\mathcal{A}| \cdot (2n-1)r!n(2n-2r)!2^r = \#(M,\sigma) \leqslant (2n)!r$$

$$|\mathcal{A}| \leqslant \frac{(2n)!r}{(2n-1)r!n(2n-2r)!2^r} = |M_n^r(e)|$$

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$$|\mathcal{A}| \leqslant \frac{1}{(r-1)!} \cdot \frac{(2n)!}{(2n-1)n(2n-2r)!2^r}$$

$$|\mathcal{A}| \leqslant \frac{1}{(r-1)!} \cdot \frac{(2n)(2n-1)\cdots(2n-2r+1)}{(2n-1)n2^r}$$

$$|\mathcal{A}| \leqslant \frac{(2n)!r}{(2n-1)r!n(2n-2r)!2^r}$$

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$$|\mathcal{A}| \leqslant \frac{1}{(r-1)!} \cdot \binom{2n-2}{2} \cdots \binom{2n-2(r-1)}{2} = |M_n^r(e)|$$

The structural claim

Let $\mathcal{A}$ be an extremal intersecting family of r-matchings of K_{2n}.

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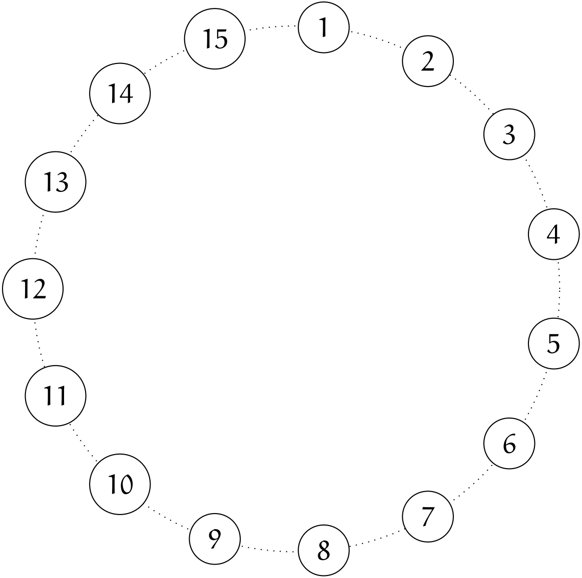
Then there must be an edge that is common to all matchings in $\mathcal{A}$.

High Level Proof Strategy

We know that if $\mathcal{A}$ is an extremal intersecting family, then:

$$|\mathcal{A}_\sigma| = r, \text{ for all } \sigma \in S_{2n}.$$

Therefore, the matchings of the subfamily $\mathcal{A}_\sigma$ necessarily have a common edge...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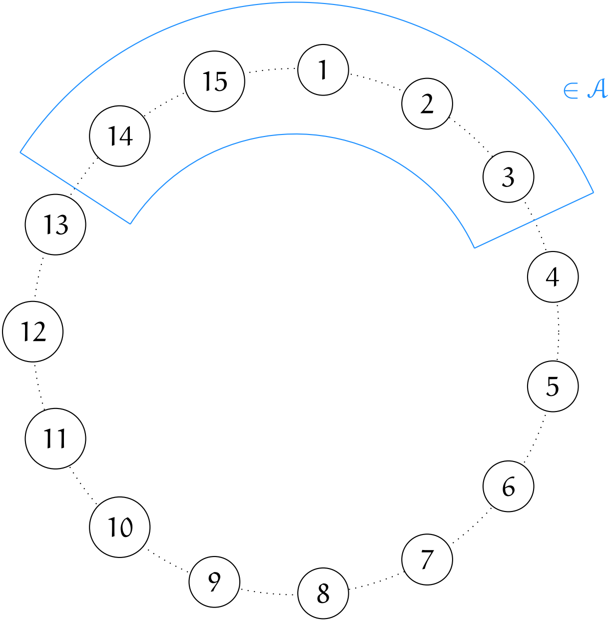

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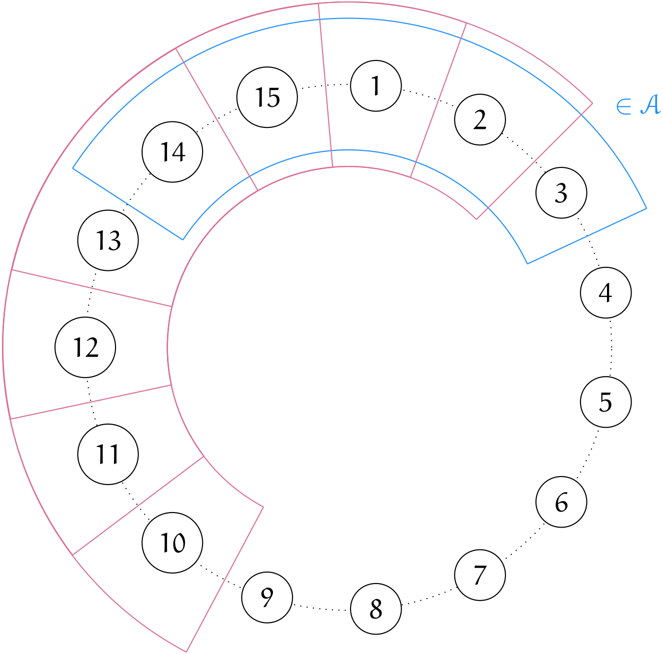

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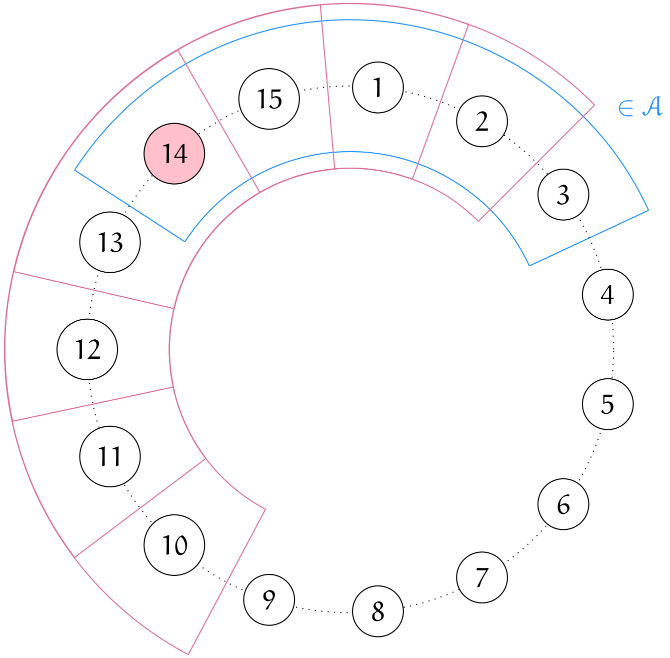

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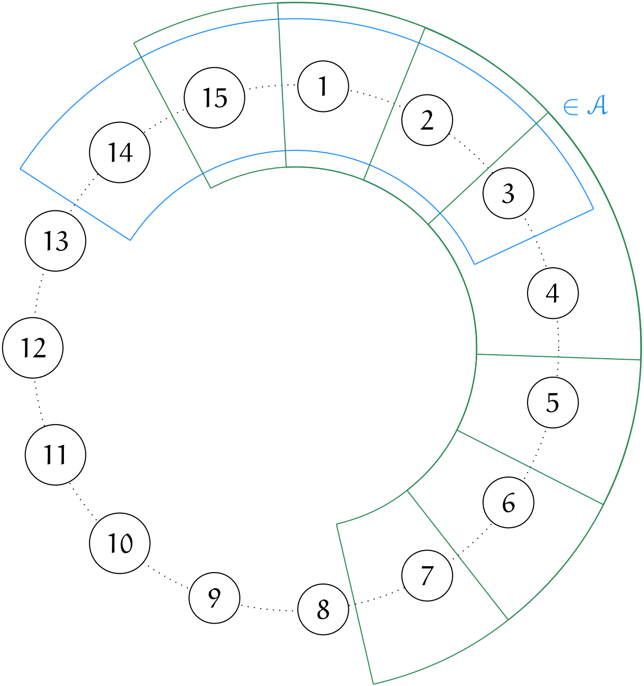

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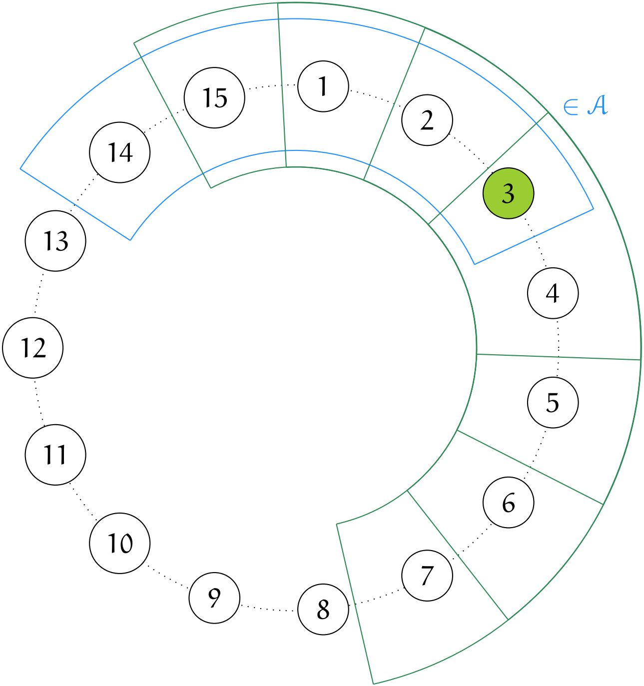

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High Level Proof Strategy (Contd.)

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it remains to show that the common edge is the same for every σ.

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High Level Proof Strategy (Contd.)

Let σ_i be obtained by σ by a transposition of the element at i.

$$\sigma_i(j) = \begin{cases} i+1 & \text{if } j = i, \\ i & \text{if } j = i+1, \\ j & \text{otherwise }. \end{cases}$$

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The proof follows by an analysis on the structure of χ_{σ_i},
based on position of i.

Thank You!