

On the Parameterized Complexity of the Maximum Edge 2-Coloring Problem

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Abstract

We investigate the parameterized complexity of the following edge coloring problem motivated by the problem of channel assignment in wireless networks. For an integer $q \geq 2$ and a graph G , the goal is to find a coloring of the edges of G with the maximum number of colors such that every vertex of the graph sees at most q colors. This problem is NP-hard for $q \geq 2$, and has been well-studied from the point of view of approximation. Our main focus is the case when $q = 2$, which is already theoretically intricate and practically relevant. We show fixed-parameter tractable algorithms for both the standard and the dual parameter, and for the latter problem, the result is based on a cubic problem kernel.

1 Introduction

Graph coloring problems are a broad and fundamental class of problems, involving an assignment of colors to the elements of a graph subject to certain constraints. They are often useful in modeling practical questions (map coloring, scheduling, register allocation, and pattern matching, to name a few), and have therefore been of central algorithmic interest. On the other hand, they have also been the subject of intensive structural study.

We are interested in the following edge coloring problem. For an integer $q \geq 2$ and a simple, undirected graph $G = (V, E)$, an assignment of colors to the edges of G is called an edge q -coloring if for every vertex $v \in V$, the edges incident on v are colored with at most q colors. An edge q -coloring that uses the maximum number of colors is called a maximum edge q -coloring. We note that the flavor of this question is quite different from the classical edge coloring question, which is a minimization problem, and the constraints require a vertex to be incident to completely distinct colors. This problem definition is motivated by the problem of channel assignment in wireless networks. The interference between the frequency channels is understood to be a bottleneck for bandwidth in wireless networks. The goal is to minimize interference to optimize bandwidth. Some wireless LAN standards allow multiple non-overlapping frequency channels to be used simultaneously. In this scenario, a computer on the network equipped with multiple interface cards can use multiple channels. The goal is to maximize the number of channels used simultaneously, if all the nodes in the network have q interface cards. It turns out that the network can be modeled by a simple, undirected graph, while the channel assignment problem corresponds to a coloring of the edges where the edges incident to any given vertex are not colored with more than q colors. The maximum edge q -coloring is also considered in combinatorics, as a particular case of the anti-Ramsey number, see [1] for the details of this formulation.

The problem is already interesting for the special case when $q = 2$. It is known to be NP-complete and APX-hard [1] and also admits a 2-approximation algorithm [10]. In their work on this problem, Feng et

al [10] show the problem to be polynomial time for trees and complete graphs for $q = 2$, and Adamaszek and Pupa [1] demonstrate a $5/3$ -approximation algorithm for graphs which have a perfect matching. Given these developments, it is natural to pursue the parameterized complexity of the problem. Our main focus will be on the case when $q = 2$, and we note that this special case continues to be relevant in practice.

The goal of parameterized complexity is to find ways of solving NP-hard problems more efficiently than brute force. Here the aim is to restrict the combinatorial explosion to a parameter that is hopefully much smaller than the input size. It is a two-dimensional generalization of “P vs. NP” where, in addition to the overall input size n , one studies how a secondary measurement (called the *parameter*), that captures additional relevant information, affects the computational complexity of the problem in question. Parameterized decision problems are defined by specifying the input, the parameter, and the question to be answered. The two-dimensional analogue of the class P is decidability within a time bound of $f(k)n^c$, where n is the total input size, k is the parameter, f is some computable function and c is a constant that does not depend on k or n . A parameterized problem that can be decided in such a time-bound is termed *fixed-parameter tractable* (FPT). For general background on the theory of fixed-parameter tractability, see [9], [11], and [16].

A parameterized problem is said to admit a *polynomial kernel* if every instance (I, k) can be reduced in polynomial time to an equivalent instance with both size and parameter value bounded by a polynomial in k . The study of kernelization is a major research frontier of parameterized complexity and many important recent advances in the area are on kernelization. These include general results showing that certain classes of parameterized problems have polynomial kernels [2, 5, 12] or randomized kernels [15]. The recent development of a framework for ruling out polynomial kernels under certain complexity-theoretic assumptions [4, 7, 13] has added a new dimension to the field and strengthened its connections to classical complexity. For overviews of kernelization we refer to surveys [3, 14] and to the corresponding chapters in books on parameterized complexity [11, 16].

Our Contributions. We develop FPT algorithms and kernels for the maximum edge 2-coloring problem. The standard parameter is the solution size, or the number of colors used. On the other hand, it is known that the maximum number of colors used in an edge 2-coloring in a graph on n vertices is at most the number of vertices in the graph. This leads to a natural “dual” parameterization below an upper bound. Specifically, we ask if we can color the graph with at least $(n - k)$ colors, and we treat k as the parameter. As an aside, we also characterize the class of graphs that can be colored with n colors as being two-factors (this is implicit in several notions in the literature, and we state the proof for completeness).

A straightforward and well-understood observation is that the maximum edge 2-coloring number is at least the size of the maximum matching of the graph. Therefore, if a graph G has a matching of size at least k , then G is a YES-instance. This is a simple polynomial time preprocessing step, and therefore we may assume throughout that the maximum matching in the input graph is bounded by k . Consequently, the vertex cover of the input is bounded by $2k$ and the treewidth is bounded by $(2k + 1)$. We do not consider these natural structural parameterizations separately, since they are implicitly bounded in terms of the solution size.

We do note that the expressibility of the maximum edge 2-colorability question in MSO_2 is easily verified. Therefore, we may easily classify the the problem as being FPT (parameterized by the solution size), by an application of Courcelle’s theorem [6]. However, we first show an exponential kernel obtained by the application of some simple reduction rules, which also implies that the problem is FPT. We then present a concrete FPT algorithm that runs in time $\mathcal{O}^*(k^k)^1$ for the problem. Also, for the dual parameterization, we obtain a polynomial instance kernel, with $\mathcal{O}(k^2)$ vertices and edges. This implies a FPT algorithm with running time $\mathcal{O}^*(k^{k^2})$.

This paper is organized as follows. In Section 2 we provide some basic definitions and facts. In section 3, we consider the standard parameter and present an exponential kernel and a FPT algorithm. In section 4, we consider the dual parameter and show a quadratic kernel.

¹The \mathcal{O}^* notation is used to suppress polynomial factors in the running time (c.f. Section 2).

2 Preliminaries

In this section we state some basic definitions related to parameterized complexity and graph theory, and give an overview of the notation used in this paper. To describe running times of algorithms we sometimes use the O^* notation. Given $f : \mathbb{N} \rightarrow \mathbb{N}$, we define $O^*(f(n))$ to be $O(f(n) \cdot p(n))$, where $p(\cdot)$ is some polynomial function. That is, the O^* notation suppresses polynomial factors in the running-time expression. The set of natural numbers (that is, nonnegative integers) is denoted by \mathbb{N} . For a natural number n let $[n] := \{1, \dots, n\}$. By $\log n$ we mean $\lceil \log n \rceil$ if an integer is expected.

Graphs In the following, let $G = (V, E)$ and $G' = (V', E')$ be graphs, and $U \subseteq V$ some subset of vertices of G . The union of graphs G and G' is defined as $G \cup G' = (V \cup V', E \cup E')$, and their intersection is defined as $G \cap G' = (V \cap V', E \cap E')$. A set U is said to be a *vertex cover* of G if every edge in G is incident to at least one vertex in U . U is said to be an *independent set* in G if no two elements of U are adjacent to each other. The *independence number* of G is the number of vertices in a largest independent set in G . U is said to be a *clique* in G if every pair of elements of U is adjacent to each other. A set U is said to be a *dominating set* in G if every vertex in $V \setminus U$ is adjacent to some vertex in U . A *two-factor* is a graph where every vertex has degree exactly two. We refer the reader to [8] for details on standard graph theoretic notation and terminology we use in the paper.

Parameterized Complexity A parameterized problem Π is a subset of $\Gamma^* \times \mathbb{N}$, where Γ is a finite alphabet. An instance of a parameterized problem is a tuple (x, k) , where k is called the parameter. A central notion in parameterized complexity is *fixed-parameter tractability (FPT)* which means, for a given instance (x, k) , decidability in time $f(k) \cdot p(|x|)$, where f is an arbitrary function of k and p is a polynomial in the input size. The notion of *kernelization* is formally defined as follows.

Definition 1. [Kernelization] [16, 11] *A kernelization algorithm for a parameterized problem $\Pi \subseteq \Gamma^* \times \mathbb{N}$ is an algorithm that, given $(x, k) \in \Gamma^* \times \mathbb{N}$, outputs, in time polynomial in $|x| + k$, a pair $(x', k') \in \Gamma^* \times \mathbb{N}$ such that (a) $(x, k) \in \Pi$ if and only if $(x', k') \in \Pi$ and (b) $|x'|, k' \leq g(k)$, where g is some computable function. The output instance x' is called the kernel, and the function g is referred to as the size of the kernel. If $g(k) = k^{O(1)}$ (resp. $g(k) = O(k)$) then we say that Π admits a polynomial (resp. linear) kernel.*

The Maximum Edge Coloring Problem Let $G = (V, E)$ be a graph, and let c be an assignment of k colors to the edges of G , that is, let c be a surjective function from E to $[k]$. We say that c is an edge coloring of the graph using k colors. For a subset F of the edge set E , let $c(F)$ denote the set of colors assigned to the edge set F , that is,

$$c(F) = \bigcup_{e \in F} c(e).$$

We say that c is *q-valid* if every vertex in the graph is incident to edges colored with at most q distinct colors. Formally, if F_v denotes the set of edges incident on a vertex v , then an edge coloring c is *q-valid* if $|c(F_v)| \leq q$ for all $v \in V$. We denote by $\sigma_q(G)$ the largest integer k for which there exists a *q-valid* edge coloring function with k colors. When considering the special case $q = 2$, we drop the subscript, and simply use $\sigma(G)$ to refer to the maximum number of colors with which G admits a 2-valid edge coloring. The first algorithmic question that arises is the following:

Max Edge 2-Coloring

Parameter: k

Input: A graph G and an integer k

Question: Is $\sigma(G) \geq k$, that is, is there a 2-valid edge coloring of G with at least k colors?

We first note that the MAX EDGE 2-COLORING problem is equivalent to its exact version:

Proposition 1. For a graph G , $\sigma(G) \geq k$ if and only if there is a 2-valid edge coloring of G with exactly k colors.

Proof. Suppose $\sigma(G) \geq k$, and let c be a 2-valid edge coloring of G using $(k + r)$ colors, $r \geq 0$. Consider the set of edges e for which $c(e) > k$, that is, consider:

$$F := c^{-1}(k + 1) \cup c^{-1}(k + 2) \cup \dots \cup c^{-1}(k + r).$$

Let c^* be the coloring that “re-colors” the edges in F with the color k , that is:

$$c^*(e) = \begin{cases} k & \text{if } e \in F \\ c(e) & \text{otherwise.} \end{cases}$$

It is easy to see that c^* is a 2-valid edge coloring that uses k colors. The other direction of the statement follows directly from the definition of $\sigma(G)$. □

Therefore, when parameterizing by the standard parameter, we will address the question of whether there is a 2-valid edge coloring that uses exactly k colors, and we refer to this as the EXACT EDGE 2-COLORING problem. We now introduce the dual parameterization. We will need some terminology first. Let G be a graph and let $c : E \rightarrow [k]$ be a 2-valid edge coloring of G with k colors. For $1 \leq i \leq k$, let F_i denote the set of edges e for which $c(e) = i$, that is, $F_i = c^{-1}(i)$. Notice that each F_i is non-empty. Fix an arbitrary edge $e_i \in F_i$, and let H be the subgraph induced by $\{e_1, \dots, e_k\}$. We call H the *character subgraph* of G . Notice that $\Delta(H) \leq 2$. It is also easy to argue that $\sigma(G) \leq |V|$ by examining the character subgraph and using the fact that it has at most $|V|$ edges (see [10]). Therefore, we may ask the following question:

<p>$(n - k)$-Edge 2-Coloring</p> <p>Input: A graph G and an integer k</p> <p>Question: Is $\sigma(G) \geq (n - k)$, that is, is there a 2-valid edge coloring of G with at least $(n - k)$ colors?</p>	<p>Parameter: k</p>
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An useful notion is that of a *palette assignment* associated with an edge coloring c . Recall that for a vertex v , we use F_v to denote the set of edges incident on v . If $c : E \rightarrow [k]$ is an edge coloring, then the palette assignment associated with c is the function c^\dagger defined as: $c^\dagger(v) = c(F_v)$. Note that in general, c^\dagger is a function from V to $2^{[k]}$, however, if c is a 2-valid coloring, then $c^\dagger : V \rightarrow \binom{[k]}{2} \cup [k] \cup \{\emptyset\}$.

We conclude this introduction to the maximum edge coloring problem with a straightforward characterization of graphs for which $\sigma(G) = |V|$.

Proposition 2. A graph $G = (V, E)$ is a two factor if, and only if, $\sigma(G) = |V|$.

Proof. Let $n := |V|$.

If G is a two-factor, then it has exactly n edges, say $\{e_1, \dots, e_n\}$. Consider the coloring $c : E \rightarrow [n]$ given by $c(e_i) = i$. Clearly, c is surjective, and since every vertex in G has degree two, any coloring is a 2-valid coloring. In particular, this shows that $\sigma(G) = n$.

On the other hand, let $\sigma(G) = n$, let $c : E \rightarrow [n]$ be a valid 2-coloring that uses n colors, and let H be a character subgraph of G . Since $|E(H)| \leq n$ and $\sigma(G) = n$, it follows that $|E(H)| = n$. Further, since $\Delta(H) \leq 2$, we also have that $|V(H)| = n$. It is also easy to observe that H is a two-factor. Indeed, let p and q be the number of vertices in H that have degree one and two in H , respectively. We know that $(p + q) = n$ (note that H has no isolated vertices) and counting the edges, we get $\frac{p + 2q}{2} = n$. This gives us $p = 0$. Thus H is a spanning two-factor in G . Observe that the palettes of vertices that are non-adjacent in H are disjoint, that is:

$$c^\dagger(u) \cap c^\dagger(v) = \emptyset \text{ if } (u, v) \notin E(H).$$

Therefore, we have that $G = H$, since any edge e that is not in H is incident on vertices whose palettes with respect to c are disjoint, implying that c cannot be extended to a valid 2-coloring of $H \cup \{e\}$. \square

3 A FPT Algorithm for Max Edge 2-Coloring

We begin by describing an exponential kernel for the EXACT EDGE 2-COLORING problem. We will subsequently describe a detailed FPT algorithm. We first observe that if G has a matching of k edges, then it is already a YES-instance of the problem.

Proposition 3. *Let (G, k) be an instance of EXACT EDGE 2-COLORING, and let m denote the number of edges in G . If $m < k$, then G is a NO-instance. If the size of the maximum matching in G is at least $(k - 1)$ and $m \geq k$, then G is a YES-instance.*

Proof. The first part of the claim is trivial: if $m < k$, then it is not possible to color the edges of G with k colors, since every edge is assigned exactly one color.

On the other hand, let $m \geq k$, and let $M = \{e_1, \dots, e_r\}$ be the set of edges in a maximum matching of G . Consider the following coloring for G :

$$c(e) = \begin{cases} i & \text{if } e = e_i \\ 0 & \text{otherwise.} \end{cases}$$

It is easily checked that the coloring c uses $(r + 1)$ colors (note that there is at least one edge that is colored 0 by c) and is 2-valid. The claim follows. \square

Let $(G = (V, E), k)$ be an instance of EXACT EDGE 2-COLORING. The first step towards an exponential kernel is to identify a matching of maximum size, say M , and return a trivial YES-instance if $|M| \geq k - 1$. If this is not the case, let $S \subseteq V$ be the set of both endpoints of every edge in M . We use I to denote $V \setminus S$, and s to denote $|S|$. Note that $s \leq 2k - 4$ and I is an independent set.

For $T \subseteq S$, let $I_T \subseteq I$ denote the set of vertices v in I for which $N(v) = T$. Note that $\{I_T \mid T \subseteq S\}$ forms a partition of I into at most 2^s classes. We are now ready to suggest our first reduction rule.

(R1) For $T \subseteq S$, and let $r := \max\{10, |T|\}$. If $|I_T| > r$, delete all but r vertices from I_T . The reduced instance has the same parameter as the original.

It is easy to see that this reduction rule may be applied $O(|I|)$ time. We now prove the correctness of this rule.

Proposition 4. *Let (G, k) be an instance of EXACT EDGE 2-COLORING, let S be a vertex cover of G and let $T \subseteq S$. Let (H, k) be the instance obtained by applying **(R1)** to G with respect to T . The instances (G, k) and (H, k) are equivalent.*

Proof. Let (G, k) be a YES-instance of EXACT EDGE 2-COLORING, and let c be a 2-valid edge coloring of G that uses k colors. Notice that since H is a subgraph of G , we may restrict c to H to obtain a 2-valid coloring of H . However, it is not clear if such a restriction would use all the k colors. It turns out that this can be ensured by a carefully planned restriction.^{II}

Consider $T \subseteq S$. If $|I_T| \leq \max\{10, |T|\}$, then there is nothing to prove. Therefore, suppose $|I_T| > \max\{10, |T|\}$, and consider the palettes of vertices of I_T with respect to c , namely, $\mathcal{P}_T := \{c^\dagger(v) \mid v \in I_T\}$. Note that, for $v \in I_T$, if $c^\dagger(v) = \{p, q\}$, then for every $u \in T$, $c^\dagger(u) \cap \{p, q\} \neq \emptyset$. We now have a case analysis on \mathcal{P}_T .

^{II}In what follows, we use the same notation for vertex subsets that are common to G and H , while ensuring that the graph in question is clear from the context.

Case 1. \mathcal{P}_T contains at least one pair of mutually disjoint sets. Note that all sets in \mathcal{P}_T have size at most two. We analyze the situation assuming that both sets have two elements, the same argument (and therefore, the same conclusion) holds for the situations when one or both sets have size one.

Let $\{p, q\} \in \mathcal{P}_T$ and $\{p', q'\} \in \mathcal{P}_T$ such that $\{p, q\} \cap \{p', q'\} = \emptyset$. Then for every $u \in T$, we have:

$$c^\dagger(u) \cap \{p, q\} \neq \emptyset \text{ and } c^\dagger(u) \cap \{p', q'\} \neq \emptyset.$$

We claim that in this situation, for any $v \in I_T$, $c^\dagger(v) \subseteq \{p, p', q, q'\}$. Indeed, suppose not, and let $i \in c^\dagger(v)$ and $i \notin \{p, p', q, q'\}$. But this would imply that $i \in c^\dagger(u)$ for some $u \in T$. However, $c^\dagger(u)$ already contains one of p or q and one of p' or q' , and $|c^\dagger(u)|$ is at most two, implying that there is no room for the additional color i — a contradiction.

Therefore, $\mathcal{P}_T \subseteq 2^Q$ where $Q := \{p, p', q, q'\}$. Noting that \mathcal{P}_T contains sets of size at most two, we have that $|\mathcal{P}_T| \leq 10$.

Case 2. \mathcal{P}_T contains no mutually disjoint sets, in other words, \mathcal{P}_T is a pairwise intersecting family. Notice that \mathcal{P}_T is a family of sets of size at most two over the universe: $\mathcal{C}_T := \bigcup_{u \in T} c^\dagger(u)$.

We first claim that $|\mathcal{C}_T| \leq |T| + 2$. To see this, consider the palettes of vertices of T with respect to c , namely, $\mathcal{Q}_T := \{c^\dagger(u) \mid u \in T\}$. Suppose the maximum number of mutually disjoint sets in \mathcal{Q}_T is two. Then $|\mathcal{C}_T| \leq 4 + (|T| - 2) = |T| + 2$. On the other hand, note that \mathcal{Q}_T does not have more than two mutually disjoint sets — indeed, having three mutually disjoint sets in \mathcal{Q}_T would mean that c cannot be extended to a 2-valid coloring of any vertex in I_T . Therefore, we have that $|\mathcal{C}_T| \leq |T| + 2$.

Now we return \mathcal{P}_T , which is an intersecting family of sets of size at most two over an universe of size at most $|T| + 2$. It follows that $|\mathcal{P}_T| \leq \max\{|T| + 1, 3\}$.

Overall, we conclude that $|\mathcal{P}_T| \leq \max\{10, |T| + 1\}$.

In I_T , we form a maximal sub-collection I'_T such that all palettes are distinct. Note that $|I'_T| = |\mathcal{P}_T|$. Let $t := |I'_T|$. Now, let J_T be the vertices in H whose neighborhood in S is T . Color the first t vertices of J_T according to I'_T and the remaining vertices arbitrarily. Note that this coloring is always possible since $|J_T| = \max\{10, |T| + 1\}$, and the first t vertices are always available. It is routine to check that the proposed coloring on H uses every color used by c in G .

In the reverse direction, let c be a 2-valid edge coloring of H that uses k colors. Let v be a vertex in G that was not affected by **(R1)**. For all edges incident on v , we simply mimic the coloring c . Let v be a vertex that was deleted according to **(R1)**. Since $v \in I_T$ for some T , there is at least one vertex $w \in I_T$ that is not affected by **(R1)**. Color the edges incident on v according to the coloring of c on the edges incident to w (note that because w and v have exactly the same neighborhood in G , this extension is 2-valid). Note that H appears as a subgraph of G , and the coloring on G restricted to this subgraph is identical to the coloring of H . It follows that all colors used by c in H are used in the coloring that we have proposed for G . This concludes the proof. □

Lemma 1. EXACT EDGE 2-COLORING has a kernel on $O(4^k \cdot (k + 1))$ vertices.

Proof. Notice that once reduced with respect to **(R1)**, for every $T \subseteq S$, there are at most $\max\{10, |T| + 1\}$ vertices in G . Thus, a conservative upper bound on the number of vertices in a reduced instance would be $(|S| + 2^s|S|)$, and the lemma follows from the fact that $s \leq 2k - 4$. □

We now turn to a FPT algorithm for EXACT EDGE 2-COLORING. See Algorithm 1 for a description of the overall algorithm. Recall that the goal is to compute a 2-valid edge coloring that uses k colors. We begin by using Proposition 3 to accept instances with a maximum matching on at least $k - 1$ edges, and reject instances that have fewer than k edges. Otherwise, let S be the vertex cover obtained by choosing both endpoints of a maximum matching.

Algorithm 1: A branching algorithm for EXACT EDGE 2-COLORING

Input: $(G = (V, E), k)$, an instance of EXACT EDGE 2-COLORING. We use $n := |V|$, $m := |E|$.

Output: YES if G has a 2-valid edge coloring that uses k colors, NO otherwise.

```
1 If  $m < k$ , return NO.
2 Compute a maximum matching  $M$  in  $G$ .
3 if  $|M| \geq k - 1$  then
4   | return YES.
5 else
6   Let  $S := V(M)$ . Let  $\mathcal{T} := \{\tau \mid \tau: S \rightarrow [k] \cup \binom{[k]}{2}\}$ .
   // For every palette assignment  $\tau$ , we check if  $\tau$  can be extended to a 2-valid
   // coloring of  $G$  that uses at least  $k$  colors.
7   for  $\tau \in \mathcal{T}$  do
   // Sanity Checks.
8     for  $e = (u, v) \in G[S]$  do
9       | if  $\tau(u) \cap \tau(v) = \emptyset$  then
10        |   Continue
11        |
12        end
13     if  $\bigcup_{u \in S} \tau(u) \neq [k]$  then
14       |   Continue
15     // Guess the colors that are realized in  $G[S]$ .
16     Let  $\mathcal{X} := 2^{[k]}$ .
17     Define  $c: E \rightarrow [k]$  as the empty coloring function, that is, for all  $e \in E$ , let  $c(e) := \emptyset$ .
18     for  $X \in \mathcal{X}$  do
19       |  $T_X := \mathbf{CheckTop}(G, k, S, \tau, c, X)$ ;
20       |  $A_X := \mathbf{CheckAcross}(G, k, S, \tau, c, [k] \setminus X)$ ;
21       | if  $(T_X = \text{YES and } A_X = \text{YES})$  then
22         |   return YES
23         |
24       end
25   end
26   return NO
27 end
```

The algorithm begins by guessing a palette assignment τ to the vertices in S . First, some simple sanity checks are implemented. Note that if c is a 2-valid edge coloring of G that uses k colors, and S is a vertex cover of G , then $\bigcup_{v \in S} c^\dagger(v) = [k]$ (if not, the missing color cannot be attributed to any edge). Therefore, we ensure that $\bigcup_{v \in S} \tau(v) = [k]$. Also, for an edge in S , the palettes assigned to the endpoints clearly cannot be disjoint. Therefore, for $u, v \in S$, if $(u, v) \in E$, we ensure that $\tau(u) \cap \tau(v) \neq \emptyset$.

Let G be a YES-instance of EXACT EDGE 2-COLORING, and suppose c is a 2-valid edge coloring of G that uses k colors. Let $X_c \subseteq [k]$ be the set of colors used by c on S . More formally, $X_c := \bigcup_{e \in G[S]} c(e)$. The second step of the algorithm involves guessing this subset of colors, that is, we consider all possible subsets of $[k]$ as candidates for being the exact set of colors that are realized by some 2-valid coloring when restricted to $G[S]$.

Let $X \subseteq [k]$ be the colors that are to be realized in S . Note that for $u, v \in S$, if $(u, v) \in E$, $p_{uv} := \tau(u) \cap \tau(v)$ either has one or two colors. If the intersection has one color, say i , and $i \notin X$, then we reject the guess X . On the other hand, if $i \in X$, we assign i to the edge between u and v and remove i from X . Notice that this is a “forced” assignment, since this is the only way to extend c to the edge uv while respecting τ . On the other hand, suppose p_{uv} has two colors. If neither of these colors is in X , then we may reject this guess. If it has two colors and only one of them is in X , then we assign the color in X to (u, v) . Otherwise, we branch on the two possibilities of $c(u, v)$, which come from p_{uv} . Note that $|X|$ drops by exactly one in both branches, so this is a two-way branching, where the corresponding search tree has depth bounded by $|X|$. This completes the description of the functionality of **CheckTop** (see also Function **CheckTop**

defined below).

Finally, we need to realize the colors in $[k] \setminus X$ on the edges that have one endpoint each in S and $G \setminus S$. To this end, we compute the lists of feasible assignments of colors for each vertex in $G \setminus S$, based on τ . In particular, a pair of colors $\{i, j\}$ belongs to the feasibility list $\ell(u)$ of a vertex $u \in G \setminus S$ if there is a way of coloring the edges incident on u with the colors i and j while respecting the palette τ . In other words, one of the colors i or j appears in $\tau(v)$ for every $v \in N(u)$. If such a list is empty, then we know that no feasible extension of τ exists. On the other hand, if the list contains a unique set, then we may color the edges incident on u according to the unique possibility.

Other than the special cases above, we know, for the same reasons as in the proof of Proposition 4, that these lists either have constant size, or have one color in common. When the lists have one color in common, then this color can be removed from $[k] \setminus X$, as such a color will be used by any coloring c that respects τ .

For lists $\ell(u)$ of constant size, as long as at least two elements in the list contain a color from $[k] \setminus X$, we branch on such elements. Note that the depth of branching is bounded by $[k] \setminus X$ and the width is bounded by a constant (at most 10, see Proposition 4). If exactly one element in $\ell(u)$ contains a color from $[k] \setminus X$, then we color u according to that element. If no elements in $\ell(u)$ contain colors from $[k] \setminus X$, then color u according to any element in the list of its feasible assignments.

Finally, we are left with a situation where some colors from $[k] \setminus X$ still need to be assigned, and the only vertices from $G \setminus S$ that are left are those whose lists contain a common color. Now this is a question of whether every color that remains in $[k] \setminus X$ can be matched to a vertex from $G \setminus S$ whose feasibility list contains that color. To this end, we construct the bipartite graph $H = ((A \cup B), E)$ as follows. The vertex set A has one vertex for every color in $[k] \setminus X$. The vertex set B has one vertex for every $u \in G \setminus S$ for which the feasibility list of u has a common color. For $i \in A$ and $u \in B$, add the edge (i, u) if $\ell(u)$ has a set which contains i . Now we compute a maximum matching M in H , and it is easy to see that the remaining colors can be realized if and only if M saturates A (see also the pseudocode for function **CheckAcross**).

This brings us to the main result of this section.

Theorem 1. *There is an algorithm with running time $\mathcal{O}^*((20k)^k)$ EXACT EDGE 2-COLORING.*

Proof. The correctness of Algorithm 1 is accounted for in the description of the algorithm.

Guessing the palette assignment requires time $\mathcal{O}^*((k + \binom{k}{2})^k)$ and guessing $X \subseteq [k]$ incurs an expense of 2^k . We note that the only branching steps happen in lines 22–28 of **CheckTop** and lines 47–52 in **CheckAcross**. The former is a two-way branching with a cost of $2^{|X|}$ and the latter is a 10-way branching with a cost of $10^{|[k] \setminus X|}$. Overall, therefore, the running time of these branching steps is bounded by 10^k . Therefore, the overall running time is bounded by $\mathcal{O}^*(20^k)$, as desired. □

4 Parameterizing Below an Upper Bound: A Quadratic Kernel

We now address the question of whether a given graph $G = (V, E)$ admits a 2-valid edge coloring using at least $(n - k)$ colors, where $n := |V|$. In this section, we show a polynomial kernel with parameter k . We note that the NP-hardness of the question is implicit in the NP-hardness of the MAX EDGE 2-COLORING PROBLEM shown in [1].

The kernel is essentially obtained by studying the structure of a YES-instances of the problem. We argue that if G is a YES-instance, c is a 2-valid edge coloring of G using at least $(n - k)$ colors, and H is a character subgraph of G with respect to c , then $|V(H)|$ must be at least $(n - k)$, or in other words, $G \setminus H$ is at most k . We then proceed to show that the components which are not cycles in H are also bounded. An easy but crucial observation is that any vertex cannot be adjacent to too many vertices whose palettes are disjoint. On the other hand, we are able to bound the number of vertices in H whose palettes are not disjoint. This leads to a bound on the maximum degree of G in terms of k . Finally, we show a reduction

rule that applies to “adjacent degree two vertices”, and this finally rounds off the analysis of the kernel size. We now formally describe the sequence of claims leading up to the kernel.

We begin by analyzing the structure of YES-instances of the problem. Let $G = (V, E)$ be a graph that admits a 2-valid edge coloring using at least $(n - k)$ colors. Let c be such a coloring, and let H be a character subgraph with respect to c . Since $\Delta(H) \leq 2$, the components of H comprise of paths and cycles. Let C_1, \dots, C_r denote the components of H that are cycles and let P_1, \dots, P_s denote the components that are paths. Let the sizes of these components be $c_1, \dots, c_r, p_1, \dots, p_s$, respectively. We first claim that $s \leq k$.

Proposition 5. *Let c be a 2-valid edge coloring of G using at least $(n - k)$ colors, and let H be a character subgraph with respect to c . If H consists of s paths of lengths p_1, \dots, p_s and r cycles of lengths c_1, \dots, c_r , then $s \leq k$.*

Proof. First, note that the number of edges in H is at least $(n - k)$. With this as a starting point, we obtain:

$$\begin{aligned} \sum_{i=1}^r c_i + \sum_{i=1}^s (p_i - 1) &\geq (n - k) \\ \underbrace{\left(\sum_{i=1}^r c_i + \sum_{i=1}^s p_i \right)}_{=|V(H)| \leq |V(G)|} - s &\geq (n - k) \\ (n - s) &\geq (n - k) \\ s &\leq k \end{aligned}$$

□

Next, we show that there are at most k vertices in G that are not in H .

Proposition 6. *Let c be a 2-valid edge coloring of G using at least $(n - k)$ colors, and let H be a character subgraph with respect to c . Then, $|G \setminus H| \leq k$.*

Proof. Suppose, for the sake of contradiction, that $|G \setminus H| > k$. This in turn implies that $|H| < n - k$. Recall, however, that $\Delta(H) \leq 2$, and therefore:

$$|E(H)| \leq \frac{2|H|}{2} = |H| < n - k.$$

However, since H is character subgraph of G with respect to a coloring that uses at least $(n - k)$ colors, we have that $|E(H)| \geq n - k$. Therefore, the above amounts to a contradiction.

□

Let \mathcal{P} denote the set of endpoints of the paths P_1, \dots, P_s . Notice that $|\mathcal{P}| \leq 2k$. Let \mathcal{T} denote the remaining vertices in H , that is, $\mathcal{T} := H \setminus \mathcal{P}$. We now claim that the maximum degree of G is bounded:

Proposition 7. *Let G be a graph that admits a 2-valid edge coloring using at least $(n - k)$ colors. Then $\Delta(G) \leq 3k + 5$.*

Proof. Let c be a 2-valid edge coloring of G using at least $(n - k)$ colors, and let H , \mathcal{P} and \mathcal{T} be defined as above.

By Propositions 5 and 6, we have that $|\mathcal{P}| + |G \setminus H| \leq 3k$. Suppose, for the sake of contradiction, that there is a vertex $u \in G$ whose degree is at least $(3k + 6)$. Clearly, u has at least six neighbors in \mathcal{T} . Since $\Delta(H[\mathcal{T}]) \leq 2$, in any subset of six vertices of \mathcal{T} , there is at least one triplet of vertices, say x, y , and z that

are mutually non-adjacent in H . By definition of H and \mathcal{T} , we know that the palettes of x , y and z with respect to c have two colors each and are mutually disjoint:

$$c^\dagger(x) \cap c^\dagger(y) = \emptyset; c^\dagger(x) \cap c^\dagger(z) = \emptyset; \text{ and } c^\dagger(y) \cap c^\dagger(z) = \emptyset,$$

$$|c^\dagger(x)| = |c^\dagger(y)| = |c^\dagger(z)| = 2.$$

Since u is adjacent to x, y and z , we conclude that there is no way to extend c to a 2-valid coloring of the edges (u, x) , (u, y) and (u, z) . Therefore, we contradict our assumption that c is a 2-valid edge coloring of G using at least $(n - k)$ colors, and conclude that the degree of every vertex in G is at most $(3k + 5)$. \square

We now state the reduction rules that define the kernelization.

(R1) If $\Delta(G) > 3k + 5$, then return a trivial NO-instance.

(R2) Let u and v be adjacent vertices with $d(u) = d(v) = 2$, and let v' be the other neighbor of v . Delete v and add the edge (u, v') . Let the graph obtained thus be denoted by H . Then the reduced instance is $(H, n^* - k)$, where $n^* = |V(H)| = (n - 1)$. Notice that the parameter does not change.

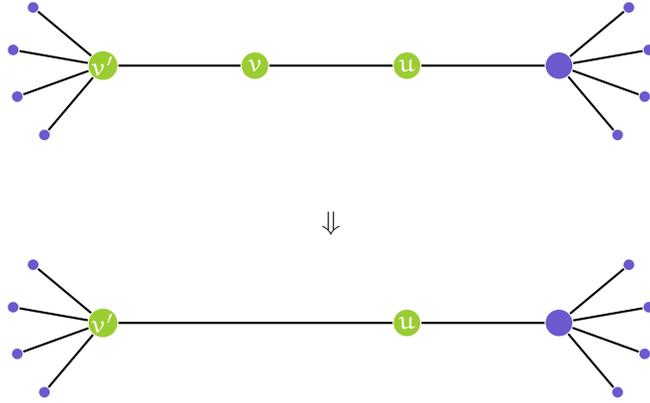


Figure 1: A reduction rule for adjacent degree two vertices.

It is easy to see that both the reduction rules above can be executed in linear time. The correctness of **(R1)** follows from Proposition 7. We now show the correctness of the second reduction rule.

Proposition 8. *Let G be a graph where vertices u and v are adjacent, and $d(u) = d(v) = 2$. Let v' be the other neighbor of v . Let H be the graph obtained from G after an application of reduction rule **(R2)**. The graph G has a 2-valid edge coloring that uses $(n - k)$ colors if and only if the graph H has a 2-valid edge coloring that uses $(n - k - 1)$ colors.*

Proof. We begin by assuming that G has a 2-valid edge coloring that uses $(n - k)$ colors. Let c be such a coloring for G . We propose the following edge coloring for H : the edge (u, v') is colored with $c(v, v')$, and all other edges are colored according to the color given by c to the corresponding edges in G . More precisely,

$$c^*(e) = \begin{cases} c(v, v') & \text{if } e = (u, v') \\ c(e) & \text{otherwise.} \end{cases}$$

Notice that c^* uses at least $(n - k - 1)$ colors, since at most one color was lost in the recoloring, namely the color given to the edge (u, v) . It is easily checked that c^* is 2-valid.

On the other hand, suppose H has a 2-valid edge coloring, c^* , that uses at least $(n-k-1)$ colors. Suppose the set of colors used is $[l]$, where $l \geq n-k-1$. Now consider the following coloring for G based on c^* :

$$c(e) = \begin{cases} c^*(u, v') & \text{if } e = (v', v) \\ l+1 & \text{if } e = (u, v) \\ c^*(e) & \text{otherwise.} \end{cases}$$

In other words, we color the edge (u, v) with a new color, and color (v, v') with the color used on the edge (u, v') , and all other edges are colored by mimicking c^* . The coloring c^* is clearly 2-valid: the only vertices that are affected are v' , v and u , but v and u are vertices of degree two and therefore any coloring is 2-valid with respect to them, and we observe that $c(F_{v'}) = c^*(F_{v'})$. Notice that c uses at least $(n-k)$ colors, one more than the number of colors used by c^* . □

Observe that Proposition 8 implies the correctness of **(R2)**. We now turn to an analysis of the size of the kernel.

Lemma 2. *If $(G, n-k)$ is a YES-instance of $(n-k)$ -EDGE 2-COLORING that is reduced with respect to **(R2)**, then $|V(G)| = O(k^3)$.*

Proof. Since G is a YES-instance, it admits a 2-valid edge coloring c using at least $(n-k)$ colors. Let H be a character subgraph with respect to c . Let C_1, \dots, C_r denote the components of H that are cycles and let P_1, \dots, P_s denote the components that are paths.

Let \mathcal{P} denote the set of endpoints of the paths P_1, \dots, P_s and let \mathcal{T} denote the remaining vertices in H , that is, $\mathcal{T} := H \setminus \mathcal{P}$. By Proposition 7, the maximum degree of G is bounded by $(3k+5)$. Every vertex in \mathcal{P} , therefore, is adjacent to at most $(3k+5)$ vertices in \mathcal{T} . Since $|\mathcal{P}| \leq 2k$ (this follows from Proposition 5), the number of vertices in \mathcal{T} that have neighbors in \mathcal{P} is therefore at most $2k(3k+5)$. Notice that all other vertices in \mathcal{T} have degree two in G . Therefore, we conclude that the number of vertices of G that have degree three or more is at most $2k(3k+5) + 2k = 2k(3k+6)$.

Since $|\mathcal{P}| \leq 2k$ and $|G \setminus H| \leq k$, it remains to bound the vertices in \mathcal{T} . Notice that the vertices of \mathcal{T} have degree two or more in G . Among them, those that have degree three or more are bounded by $2k(3k+6)$. The vertices left are vertices of \mathcal{T} that have degree two. Since the graph is reduced with respect to **(R2)**, the neighbors of these vertices have degree one or degree three or more. Since the number of vertices having degree three or more is bounded by $2k(3k+6)$ and $\Delta(G) \leq 3k+5$, the total number of degree two vertices that have at least one degree three neighbor is at most $2k(3k+6)(3k+5)$. Further, note that the number of vertices of degree one is at most $|\mathcal{P}| + |G \setminus H| \leq 3k$. The number of degree two vertices for which both endpoints are vertices of degree one is at most $3k/2$. Thus the total number of vertices of degree two in \mathcal{T} is at $O(k^3)$. This concludes our argument. □

5 Concluding Remarks and Future Work

The most natural unresolved question is to settle the kernelization complexity of the maximum edge 2-coloring problem when parameterized by the solution size. The exponential kernel described in this work implies a polynomial kernel when the input is restricted to graphs where the maximum degree is a constant, and also if the input is restricted to graphs without cycles of length four. These observations are interesting because the problem continues to be NP-complete for both of these graph classes. The NP-hardness for graphs of bounded degree can be obtained by easy modifications to the reduction proposed in [1], and the NP-hardness on graphs without cycles of length four is given in the appendix for completeness. Given these results, the question of whether the problem admits a polynomial kernel on general graphs is an interesting open problem.

Improved FPT algorithms for both the standard and the dual parameter, specifically with running time $O(c^k)$ for some constant c , will be of interest as well. It is also natural to pursue the above-guarantee version of the question, with the size of the maximum matching used as the guarantee. In particular, if γ is the size of a maximum matching in a graph G , we would like to study the question of checking if G can be colored with at least $(\gamma + k)$ colors, parameterized by k .

For the more general question of maximum edge 2 coloring, note that since the problem is NP-complete for fixed values of q , the question is para-NP-complete when parameterized by q alone. Generalizing some of the results that hold for $q = 2$ is an interesting direction for future work.

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6 Appendix: NP-hardness on C_4 -free graphs

In this section, we show the NP-hardness of MAX EDGE 2-COLORING on graphs that have no cycles of length four. We will first need to introduce some definitions. For a function $f : V(G) \rightarrow \{1, 2\}$, we say that G is f -valid if every vertex v in the graph is incident to edges colored with at most $f(v)$ distinct colors. We will consider the following variant of the original problem:

Max Edge $\{1, 2\}$ -Coloring

Parameter: k

Input: A graph G , a function $f : V(G) \rightarrow \{1, 2\}$, and an integer k
 Question: Is $\sigma(G) \geq k$, that is, is there a f -valid edge coloring of G with at least k colors?

It is known from [?] that there is a polynomial time reduction from MAX EDGE $\{1, 2\}$ -COLORING to MAX EDGE 2-COLORING. If G is an instance of MAX EDGE $\{1, 2\}$ -COLORING, the reduction proposed in [1] modifies G by adding a pendant neighbor to every vertex v for which $f(v) = 1$. It is shown that this graph has a 2-valid coloring using at least k colors if, and only if, G has a f -valid coloring using at least k colors.

Proposition 9 (Theorem 5, [1]). *There is a polynomial time reduction from MAX EDGE $\{1, 2\}$ -COLORING to MAX EDGE 2-COLORING.*

We will rely on this reduction to obtain our NP-hardness results, as it simplifies our presentation greatly when we reduce to MAX EDGE $\{1, 2\}$ -COLORING instead of the original question.

We reduce from the MULTI-COLORED INDEPENDENT SET problem, which is known to be NP-complete by a reduction from the classical NP-complete problem of finding the largest independent set. The “multi-colored” variant is defined as follows:

Multi-Colored Independent Set

Parameter: k

Input: A graph $G = (V, E)$, and a partition of V into k parts, $V = V_1 \uplus \dots \uplus V_k$
 Question: Does there exist a subset $S \subseteq V$ such that $G[S]$ forms an independent set and $|S \cap V_i| = 1$, for all $1 \leq i \leq k$?

We first describe the construction. Let $G = (V, E)$, and $V = V_1 \uplus \dots \uplus V_k$ be an instance of MULTI-COLORED INDEPENDENT SET. Let (H, f, l) denote the reduced instance; we will first describe H , then determine f and finally fix l .

To begin with, the vertex set of H contains one vertex for every vertex in G . If $v \in G$, then we abuse notation and denote the corresponding vertex in H by v as well. We also add the vertices $\{g_1, \dots, g_k\}$ and make g_i a common neighbor of all vertices in V_i . In other words, we introduce the edges $\{(g_i, v) \mid v \in V_i\}$ for all $1 \leq i \leq k$. Then introduce a vertex g which is adjacent to every g_i . For every $e \in E(G)$, where $e = (u, v)$, introduce five new vertices $\{e_u, e'_u, e_3, e'_v, e_v\}$, and add the edges: $(u, e_u), (g, e_3), (v, e_v), (e_u, e'_u), (e'_u, e_3), (e_3, e'_v)$ and (e'_v, e_v) . This completes the description of H (see also Figure 6).

We define f as follows. For any $e \in G$, $f(e_3) = 2$. Also, $f(g_i) = 2$ for all $1 \leq i \leq k$. For any other vertex $v \in H$, we have $f(v) = 1$. Let $l := k + 1$. We are now ready to show the equivalence of $(G, V_1 \uplus \dots \uplus V_k, k)$ and (H, f, l) .

Lemma 3. *The graph G with vertex partitions V_1, \dots, V_k has a multi-colored independent set of size k if and only if H has a f -valid coloring that uses at least $k + 1$ colors.*

Proof. In the forward direction, let $S = \{u_1, \dots, u_k\}$ be a multi-colored independent set. Without loss of generality, let $u_i \in V_i$. Consider the following coloring function (the notation “ \star ” is used to denote any vertex):

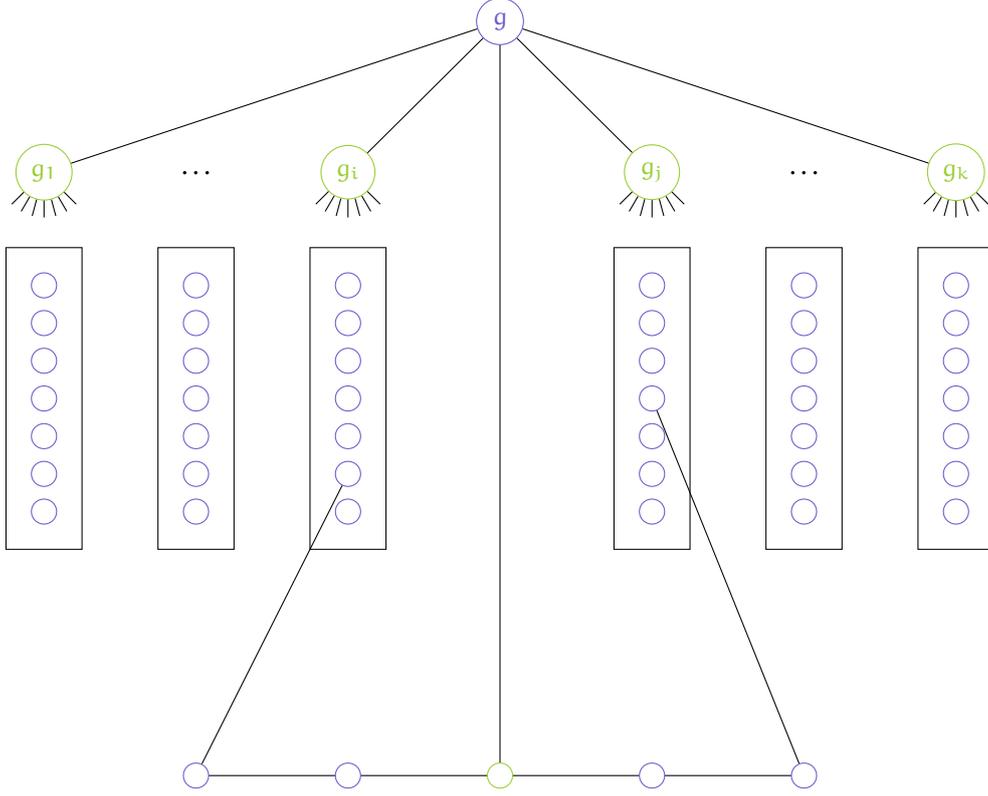


Figure 2: A description of the reduction from MULTI-COLORED INDEPENDENT SET to MAX EDGE $\{1,2\}$ -COLORING.

$$c(t) = \begin{cases} i & \text{if } t = (\star, u_i), \text{ or } t \in \{(u_i, e_{u_i}), (e_{u_i}, e'_{u_i}), (e'_{u_i}, e_3) \mid e \in E(G)\} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that c uses $(k+1)$ colors. We now argue that c is f -valid. Recall that we use F_v to denote the set of edges incident on a vertex v , and $c(F)$ to denote the set of colors assigned to the edge set F . First, observe that $c(F_g) = \{0\}$. Further, for every $u \in G$, we have either that $c(F_u) = \{i\}$ (if $u \in V_i \cap S$), or $c(F_u) = \{0\}$. Now consider the vertex g_i for any $1 \leq i \leq k$. Clearly, $N(g_i) \cap \{u_1, \dots, u_k\} = \{u_i\}$. Therefore, $c(F_{g_i}) = \{0, i\}$. Recall that c is f -valid since $f(g_i) = 2$ for all $1 \leq i \leq k$.

Finally, for $e \in G$, and $e = (u, v)$, consider the vertices e_u, e'_u, e_3, e'_v , and e_v . If $u \notin S$ and $v \notin S$, then it is easy to verify that

$$c(F_{e_u}) = c(F_{e'_u}) = c(F_{e_3}) = c(F_{e'_v}) = c(F_{e_v}) = \{0\}.$$

The other case is that one of u or v belong to S (indeed, since $G[S]$ is an independent set, S does not contain *both* u and v). Without loss of generality, let $u \in S$ and suppose $u \in V_i$. Then note that

$$c(F_{e_u}) = c(F_{e'_u}) = \{i\}, c(F_{e_3}) = \{0, i\}, \text{ and } c(F_{e'_v}) = c(F_{e_v}) = \{0\}.$$

Recall that $f(e_3) = 2$, and therefore, the above is a f -valid coloring. This completes the argument in the forward direction.

We now turn to the reverse direction. Let c be a coloring of H that uses $(k+1)$ colors, $\{0, 1, \dots, k\}$. Recall that $f(g) = 1$, and let 0 denote the unique color in $c(F_g)$. We claim that:

$$c := \bigcup_{1 \leq i \leq k} c(F_{g_i}) = [k] \cup \{0\}.$$

The argument is by contradiction. Suppose the above does not hold, and let j be the smallest color that is missing from \mathcal{C} . Let e be the edge for which $c(e) = j$ (there must be such an edge, since c uses all the $(k+1)$ colors). The edge cannot be incident to any of the g_i 's by assumption. Therefore, it must be one of the edges: $(e_u, e'_u), (e'_u, e_3), (e_3, e'_v)$ or (e'_v, e_v) , for some $e = (u, v) \in E(G)$. Now, notice that if $c(e'_u, e_3) = j$, then $c(e_u, e'_u) = j$, since $f(e'_u) = 1$. Similarly, since $f(e_u) = 1$, we have that $c(e_u, u) = j$. Finally, since $c(u) = 1$, $c(u, g_i) = j$ for some $i \in [k]$. This contradicts our assumption that there is no g_i for which $j \in c(F_{g_i})$. An analogous argument can be made if $c(e'_v, e_3) = j$, $c(e'_u, e_u) = j$, or $c(e'_v, e_v) = j$.

Therefore, we have that $c(F_{g_i}) = \{a_i, 0\}$, where $a_i \in [k]$. Notice that $a_i \neq a_j$ for any $i \neq j$ (since $\{a_1, \dots, a_k\} = [k]$). For $1 \leq i \leq k$, let $u_i \in V_i$ be a vertex for which $c(g_i, u_i) = a_i$. Notice that there is at least one such vertex, since all the neighbors of g_i are in $V_i \cup \{g\}$, and one of the edges incident on g_i is colored with a_i (note that this edge cannot be (g, g_i) , which we already know has color 0 according to c). Consider S defined as $\{u_1, \dots, u_k\}$. By definition, we have that $|S \cap V_i| = 1$ for all $1 \leq i \leq k$. We now argue that $G[S]$ is an independent set.

Suppose not, and let $e := (u_p, u_q) \in E$, for $1 \leq p \neq q \leq k$. By the definition of S , we have that $c(g_p, u_p) = a_p$ and $c(g_q, u_q) = a_q$, where $a_p \neq a_q \neq 0$. Since $c(u_p) = c(u_q) = 1$, observe that $c(u_p, e_{u_p}) = a_p$ and $c(u_q, e_{u_q}) = a_q$. Further, since $c(e_{u_p}) = c(e_{u_q}) = 1$, again, $c(e_{u_p}, e'_{u_p}) = a_p$ and $c(e_{u_q}, e'_{u_q}) = a_q$. Finally, since $c(e'_{u_p}) = c(e'_{u_q}) = 1$, we obtain:

$$c(e'_{u_p}, e_3) = a_p \text{ and } c(e'_{u_q}, e_3) = a_q,$$

Recall that $(e_3, g) \in E(H)$ and therefore, $0 \in c(F_{e_3})$. Putting this together with the above, we see that $c(F_{e_3}) = \{0, a_p, a_q\}$, contradicting our assumption that c was f -valid. \square

It follows from the construction that H as defined above has no cycles of length four.

Observe that adding pendant vertices does not add any cycles to the graph. Therefore, we have the following result by Proposition 9, Lemma 3.

Theorem 2. *The MAX EDGE 2-COLORING problem is NP-complete on graphs that have no cycles of length four.*

As an aside, we note that the reduction in this section also shows the $W[1]$ -hardness of the Max Edge $\{1, 2\}$ -Coloring variant. The fact that MAX EDGE 2-COLORING is FPT is not surprising, since the reduction from MAX EDGE $\{1, 2\}$ -COLORING to MAX EDGE 2-COLORING is not parameter-preserving.

Corollary 1. *The MAX EDGE $\{1, 2\}$ -COLORING problem is $W[1]$ -hard on graphs that have no cycles of length four.*

Function CheckTop(G, k, S, τ, c, X)

Input: G, k, S, τ, c, X , an instance of EXACT EDGE 2-COLORING, with a partial edge coloring c , a vertex cover S , palette assignment τ to vertices in S , and a subset of colors X .

Output: YES if G has a 2-valid edge coloring that uses exactly the colors in X in $G[S]$, while extending c and respecting τ , NO otherwise.

```
1 if  $X = \emptyset$  then
2   | return YES
3 for  $(u, v) \in G[S]$  and  $(u, v) \in E$  do
4   | if  $\tau(u) \cap \tau(v) = \{i\}$  then
5     |   if  $i \notin X$  then
6       |     | return NO
7       |     |  $c(u, v) := i$ 
8       |     |  $X \leftarrow X \setminus \{i\}$ 
9       |     | Continue
10  |
11 end
12 for  $(u, v) \in G[S]$  and  $(u, v) \in E$  do
13   | if  $\tau(u) = \tau(v) = \{i, j\}$  and  $|\{i, j\} \cap X| = \emptyset$  then
14     |   | return NO
15     | else if  $\tau(u) = \tau(v) = \{i, j\}$  and  $|\{i, j\} \cap X| = 1$  then
16       |   |  $\ell := \{i, j\} \cap X$ 
17       |   |  $c(u, v) := \ell$ 
18       |   |  $X \leftarrow X \setminus \ell$ 
19       |   | Continue
20     | end
21     | else
22       |   |  $c_i(u, v) := i$ 
23       |   |  $X_i \leftarrow X \setminus i$ 
24       |   |  $a = \text{CheckTop}(G, k, S, \tau, c_i, X_i);$ 
25       |   |  $c_j(u, v) := j$ 
26       |   |  $X_j \leftarrow X \setminus j$ 
27       |   |  $b = \text{CheckTop}(G, k, S, \tau, c_j, X_j);$ 
28       |   | return  $a$  or  $b$ 
29     | end
30 end
31 if  $X \neq \emptyset$  then
32   | return NO
33 end
```

Function CheckAcross(G, k, S, τ, c, X)

Input: G, k, S, τ, c, X , an instance of EXACT EDGE 2-COLORING, with a partial edge coloring c , a vertex cover S , palette assignment τ to vertices in S , and a subset of colors X .

Output: YES if G has a 2-valid edge coloring that uses the colors in X in $G \setminus E(S)$, while extending c and respecting τ , NO otherwise.

```

1  if  $X = \emptyset$  then
2  |   return YES
3  for  $u \in G \setminus S$  do
4  |    $\ell(u) \leftarrow \{\emptyset\}$ 
5  |    $\mathcal{P}(u) = \bigcup_{v \in N(u)} (\tau(v)) \subseteq [k]$ 
6  |   for  $Y \in \mathcal{P} \cup \binom{\mathcal{P}(u)}{2}$  do
7  |       for  $v \in N(u)$  do
8  |           if  $\tau(v) \cap Y \neq \emptyset$  then
9  |               |    $\ell(u) \leftarrow \ell(u) \cup Y$ 
10 |           |
11 |       end
12 |   end
13 end
14 for  $u \in G \setminus S$  do
15 |   if  $\ell(u) = \emptyset$  then
16 |       |   return NO
17 |   if  $|\ell(u)| = 1$  then
18 |       |    $Z \in \ell(u)$ 
19 |       |   for  $v \in N(u)$  do
20 |           |    $c(u, v) = \tau(v) \cap Z$ 
21 |       |   end
22 |       |    $X \leftarrow X \setminus Z$ 
23 |       |   Continue
24 |
25 |   if  $\bigcap_{Y \in \ell(u)} Y \neq \emptyset$  then
26 |       |    $i := \bigcap_{Y \in \ell(u)} Y$ 
27 |       |    $X \leftarrow X \setminus \{i\}$ 
28 |       |    $g(u) := 1$ 
29 |   else
30 |       |    $g(u) := 0$ 
31 |   end
32 end
33  $\mathcal{B} := \{u \in G \setminus S \mid g(u) = 0\}$ 
34  $\mathcal{G} := \{u \in G \setminus S \mid g(u) = 1\}$ 
35 for  $u \in \mathcal{B}$  do
36 |    $\ell^* = \emptyset$ 
37 |   for  $Y \in \ell(u)$  do
38 |       |   if  $Y \cap X \neq \emptyset$  then
39 |           |   |    $\ell^* \leftarrow \ell^* \cup Y$ 
40 |       |
41 |   end
42 |   if  $\ell^*(u) = \emptyset$  then
43 |       |   Continue
44 |   if  $|\ell^*(u)| = 1$  then
45 |       |   Color  $u$  according to the unique element in
46 |       |    $\ell^*(u)$ .
47 |       |   Continue
48 |   for  $Y \in \ell^*(u)$  do
49 |       |   Color  $u$  according to  $Y$ .
50 |       |   Let  $c'$  be the new coloring.
51 |       |    $X' \leftarrow X \setminus Y$ .
52 |       |    $a_Y = \text{CheckAcross}(G, k, S, \tau, c', X')$ 
53 |   end
54 |   return  $\max\{a_Y \mid Y \in \ell^*(u)\}$ .
55 end
56 if  $X = \emptyset$  then
57 |   return YES
58 for  $u \in \mathcal{G}$  do
59 |   Construct the bipartite graph  $H = ((A \cup B), E)$ 
60 |   as follows. The vertex set  $A$  has one vertex for
61 |   every color in  $X$ . The vertex set  $B$  has one
62 |   vertex for every  $u \in \mathcal{G}$ . For  $i \in A$  and  $v \in B$ ,
63 |   add the edge  $(i, v)$  if  $\ell(u)$  has a set which
64 |   contains  $i$ .
65 |   Compute a maximum matching  $M$  in  $H$ .
66 |   if  $M$  saturates  $A$  then
67 |       |   return YES
68 |   else
69 |       |   return NO
70 |   end
71 end

```
