Quadratic Upper Bounds on the Erdős–Pósa property for a generalization of Packing and Covering cycles

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Abstract

According to the classical Erdős–Pósa theorem, given a positive integer k, every graph G either contains k vertex disjoint cycles or a set of at most $\mathcal{O}(k \log k)$ vertices that hits all its cycles. Robertson and Seymour [Graph minors. V. Excluding a planar graph. *J. Comb. Theory Series B*, 41:92–114, 1986] generalized this result in the best possible way. More specifically, they showed that if $\mathcal{H}$ is the class of all graphs that can be contracted to a fixed planar graph H, then every graph G either contains a set of k vertex-disjoint subgraphs of G, such that each of these subgraphs is isomorphic to some graph in $\mathcal{H}$ or there exists a set S of at most $f(k)$ vertices such that $G \setminus S$ contains no subgraph isomorphic to any graph in $\mathcal{H}$. However the function f is exponential. In this note, we prove that this function becomes quadratic when $\mathcal{H}$ consists all graphs that can be contracted to a fixed planar graph θ_c. For a fixed c, θ_c is the graph with two vertices and $c \geq 1$ parallel edges. Observe that for $c = 2$ this corresponds to classical Erdős–Pósa theorem.

1 Introduction

Given a graph G we denote by $V(G)$ and $E(G)$ its vertex and edge set respectively. Let G be a graph, and let $\mathcal{H}$ be a class of graphs. The $\mathcal{H}$-PACKING problem asks for a set of vertex-disjoint subgraphs of G, called an *$\mathcal{H}$-packing*, such that each of these subgraphs is isomorphic to some graph in $\mathcal{H}$. A related problem is the $\mathcal{H}$-COVERING problem, where the question is to find a set $S \subseteq V(G)$ of vertices, an *$\mathcal{H}$-cover*, such that $G \setminus S$ contains no subgraph isomorphic to any graph in $\mathcal{H}$. The class $\mathcal{H}$ is said to have the Erdős-Pósa property for some graph class $\mathcal{G}$ if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every $k \geq 0$, every graph $G \in \mathcal{G}$ either contains an $\mathcal{H}$-packing of size at least k, or has an $\mathcal{H}$-cover of size at most $f(k)$.

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Erdős and Pósa [7] proved that the Erdős-Pósa property holds for all graphs when $\mathcal{H}$ is the class of all cycles. The problem of identifying more general graph classes where the Erdős-Pósa property is satisfied has attracted a lot of attention [2,4,11,13,17,18]. Extensions of this problem defined on matroids have also been investigated [9,10].

The operation of *contracting* an edge $e = (u, v)$ in a graph G results in a graph G', in which u and v are replaced by a new vertex v_e and in which for every neighbour w of u or v in G, there is an edge (w, v_e) in G'. We say that a graph G *can be contracted to a graph H* if H can be obtained from G by a series of edge contractions . We say that H *is a minor of* G if some subgraph $\tilde{G}$ of G can be contracted to H; such a $\tilde{G}$ is called an *H-minor model* of G. A graph class $\mathcal{G}$ is *minor-closed* if any minor of a graph in $\mathcal{G}$ is again a member of $\mathcal{G}$.

For H a fixed connected graph, the class $\mathcal{H} = \mathcal{M}H$ consists of graphs that contain H as a minor. For a fixed c, let θ_c be the graph with two vertices and $c \geq 1$ parallel edges. Observe that for $H = \theta_1$, and $H = \theta_2$, $\mathcal{H} = \mathcal{M}H$ consists of all graphs that contain at least one edge and all graphs that contain at least one cycle, respectively. Robertson and Seymour [15, Proposition 8.2] proved the following seminal result.

Proposition 1. *Let H be a connected graph. Then $\mathcal{M}H$ satisfies the Erdős-Pósa property for all graphs if and only if H is planar.*

For an alternate proof of Proposition 1, see the monograph "Graph Theory" by R. Diestel [5, Corollary 12.4.10 and Exercise 39] . The bounding function $f(k)$ in the Erdős-Pósa property, as obtained in the proof of Proposition 1, is exponential in k. Fomin et al. [8] showed that the bound becomes linear for any planar graph H when the graph class $\mathcal{G}$ is any non trivial minor-closed class. However, by the classical result of Erdős-Pósa [7], the class $\mathcal{H} = \mathcal{M}\theta_2$ has the Erdős-Pósa property with $f(k) = O(k \log k)$ when $\mathcal{G}$ is the set of all graphs. In this note we prove a quadratic bound for the case when $\mathcal{G}$ consists of all graphs and $\mathcal{H}$ consists of all graphs which can be contracted to a fixed planar graph θ_c. Observe that for $c = 2$ this corresponds to classical Erdős–Pósa theorem, albeit with a larger bound. The main result of this paper is:

Theorem 1. [Erdős-Pósa property for θ_c] *For any fixed $c \in \mathbb{N}$, every graph G either contains k vertex-disjoint θ_c-minor models, or has a θ_c-hitting-set of size at most $f(k) = O(k^2)$.*

Given a graph G and a vertex subset $S \subseteq V(G)$, we call a set S a *θ_c-hitting set* if $G \setminus S$ does not contain θ_c as a minor. In the rest of this note we use the term "hitting set" to refer to a θ_c-hitting set.

2 The Erdős-Pósa Property for θ_c

In this section we give the proof of Theorem 1. Towards this we need following definitions.

Let G be a graph. A *tree decomposition* of a graph G is a pair $(T, \mathcal{X} = \{X_t\}_{t\in V(T)})$ such that

- $\cup_{t\in V(T)} X_t = V(G)$,
- for every edge $(x, y) \in E(G)$ there is a $t \in V(T)$ such that $x, y \subseteq X_t$, and
- for every vertex $v \in V(G)$ the subgraph of T induced by the set $\{t \mid v \in X_t\}$ is connected.

The *width* of a tree decomposition is $\left(\max_{t\in V(T)} |X_t|\right) - 1$ and the *treewidth* of G is the minimum width over all tree decompositions of G. A tree decomposition $(T, \mathcal{X})$ is called a *nice tree decomposition* if T is a tree rooted at some node r where $X_r = \emptyset$, each node of T has at most two children, and each node is of one of the following kinds:

1. *Introduce node*: a node t that has only one child t' where $X_t \supset X_{t'}$ and $|X_t| = |X_{t'}| + 1$.
2. *Forget node*: a node t that has only one child t' where $X_t \subset X_{t'}$ and $|X_t| = |X_{t'}| - 1$.
3. *Join node*: a node t with two children t_1 and t_2 such that $X_t = X_{t_1} = X_{t_2}$.
4. *Base node*: a node t that is a leaf of t, is different than the root, and $X_t = \emptyset$.

Notice that, according to the above definition, the root r of T is either a forget node or a join node. It is well known that any tree decomposition of G can be transformed into a nice tree decomposition in time $O(|V(G)| + |E(G)|)$ maintaining the same width [12]. We use G_t to denote the graph induced on the vertices $\cup_{t'} X'_t$, where t' ranges over all descendants of t, including t. We use H_t to denote $G_t[V(G_t) \setminus X_t]$.

We prove Theorem 1 by establishing the following two lemmas.

Lemma 1. *If the treewidth of a graph G is at least $2c^2k^2$, then G contains at least k vertex-disjoint θ_c-minor-models.*

Lemma 2. *If the treewidth of G is at most $2c^2k^2$ and G does not contain k vertex-disjoint θ_c-minor-models, then G contains a θ_c-hitting set of size at most $\eta k^2 = O(k^2)$, where the constant η depends only on c.*

The proof of Theorem 1 follows from the above two lemmas.

Proof of Theorem 1. Suppose graph G does not contain k vertex-disjoint θ_c-minor-models. Then by Lemma 1, G has treewidth at most $2c^2k^2$. Now by applying Lemma 2, we have that G contains a θ_c-hitting set of size $O(k^2)$. □

We now define some terms which we use in the proof of Lemma 1. A *bramble* is a set of connected subgraphs, called the *elements* of the bramble, any two of which either intersect or are linked by at least one edge. A *hitting set* of a bramble is a set of vertices which meets every element of the bramble. The *order* of a bramble is the minimum cardinality of a hitting set of the bramble. The maximum order of a bramble in a graph is its *bramble number*. Brambles and tree decompositions are dual structures in the following sense.

Proposition 2. [16] *The tree-width of any graph is exactly one less than its bramble number.*

Our proof of Lemma 1 uses some ideas from the proof of Lemma 3.2 in Wood and Reed's recent work [14] on grid-like minors.

Lemma 3. [3] *Let $\mathcal{B}$ be a bramble in a graph G. Then G contains a path that intersects every element of $\mathcal{B}$.*

Now we are ready to give a proof of Lemma 1.

Proof of Lemma 1. We show that if the treewidth of a graph G is at least $2c^2k^2$, then G contains at least k vertex-disjoint θ_c-minor-models. If the treewidth of G is at least $2c^2k^2$, then by Proposition 2, G contains a bramble (call it $\mathcal{B}$) of order at least $2c^2k^2 + 1$. By Lemma 3, there exists a path that visits every element of the bramble at least once. Let P be such a path, and let $v_1, \ldots, v_t$ be the vertices of P (stated in the order of their appearance in P). Note that $t \geq 2c^2k^2 + 1$, as otherwise P would be a hitting set of $\mathcal{B}$ with fewer vertices than the order of $\mathcal{B}$.

For $1 \leq i \leq t$, let B_i denote the set of all elements of $\mathcal{B}$ which contain the vertex v_i. Note that for $1 \leq i \leq t$, $\cup_{j=1}^{i} B_j$ is a bramble. Let O_i denote the order of this bramble. Let s be the smallest number such that $O_s = c^2k^2$. The existence of such s is guaranteed by the fact that $O_1 = 1$, $O_t > 2c^2k^2$, and for $1 \leq i \leq t-1$, $O_{i+1} \leq O_i + 1$. Let $\mathcal{B}_1 = \cup_{i=1}^{s} B_i$, and let $\mathcal{B}_2 = \mathcal{B} \setminus \mathcal{B}_1$. Since the value of O_i increases by at most one in a single step, we have that the order of $\mathcal{B}_2$ is at least c^2k^2, or else the union of the smallest hitting sets for $\mathcal{B}_1$ and for $\mathcal{B}_2$ would be a hitting set of $\mathcal{B}$ which has fewer vertices than the order of $\mathcal{B}$. Let P_1 be the subpath of P starting at v_1 and ending at v_s, and P_2 the subpath starting at v_{s+1} and ending at v_t. By the above argument, P_1 and P_2 contain at least c^2k^2 vertices each.

Now, there must exist a collection, say $\mathcal{P}$, of at least c^2k^2 vertex-disjoint paths that begin in P_1 and end in P_2. If not, then by Menger's theorem, there exists a P_1–P_2 separator, say S, of size less than c^2k^2. Note that S cannot be a hitting set of the brambles $\mathcal{B}_1$ or $\mathcal{B}_2$, since the order of each of these is at least c^2k^2. So there exist elements $A \in \mathcal{B}_1, B \in \mathcal{B}_2$ such that $A \cap S = \emptyset = B \cap S$. But since A and B are connected subgraphs which either intersect or are linked by an edge — being elements of $\mathcal{B}$ — and $A \cap P_1 \neq \emptyset, B \cap P_2 \neq \emptyset$, S cannot be a P_1–P_2 separator.

We now show that $\mathcal{P} \cup P_1 \cup P_2$ contains k vertex-disjoint θ_c minor-models. Let E_p be the set of vertices that form the end points (on P_1 and P_2) of the paths in $\mathcal{P}$. For $i \in \{1, 2\}$, let $Q_i = P_i \cap E_p$. We label both Q_1 and Q_2 with a common index set $[M]$, where $M = |Q_1| = |Q_2|$. Let $f : [M] \to [M]$ be the following bijection: $f(i) = j$ if and only if there is a path in $\mathcal{P}$ that begins in i and ends in j. We say that a subset of paths $C \subseteq \mathcal{P}$ is *cross-free* under this labeling if there does not exist $i, i' \in Q_1 \cap C; i < i'$ and $f(i) > f(i')$.

Note that since the paths in $\mathcal{P}$ are vertex-disjoint, the numbers $f(1)$, $f(2)$, …, $f(M)$ form a permutation of M, and by the Erdős-Szekeres Theorem [6], the sequence $\langle f(1), f(2), \ldots, f(M) \rangle$ contains a monotonically increasing or decreasing subsequence of length at least t, where t is $\sqrt{|M|} = ck$. Let a witness subsequence be $\langle f(s_1), f(s_2), \ldots, f(s_t)\rangle$. Let $Q'_1 = \{s_1, s_2, \ldots, s_t\}$ and $Q'_2 = \{f(s_1), f(s_2), \ldots, f(s_t)\}$. Then the paths in $\mathcal{P}$ that have their end points in Q'_1, Q'_2 form a cross-free collection. These paths together with P_1, P_2 contain at least k vertex-disjoint θ_c minor-models. □

We first define the notion of a *good labeling function*. Given a nice tree decomposition $(T, \mathcal{X} = \{X_t\}_{t\in V(T)})$ of a graph G, a function $g: V(T) \rightarrow \mathbb{N}$ is called a *good labeling function* if it satisfies the following properties:

- if t is a base node then $g(t) = 0$;
- if t is an introduce node, then $g(t) = g(s)$, where s is the child of t;
- if t is a join node, then $g(t) = g(s_1) + g(s_2)$, where s_1 and s_2 are the children of t; and
- if t is a forget node, then $g(t) \in \{g(s), g(s) + 1\}$, where s is the child of t.

A *max labeling function* g is defined analogously to a good labeling function, the only difference being that for a join node t, we have the condition $g(t) = \max\{g(s_1), g(s_2)\}$. Now we are ready to prove the covering lemma—Lemma 2.

Proof of Lemma 2. In this section, we show that if G has treewidth at most $2c^2k^2$ and does not have more than $k' = k - 1$ disjoint minor-models of θ_c, then there exists a set $S \subseteq V(G)$, $|S| = O(k^2)$, such that $G \setminus S$ does not contain θ_c as a minor.

Consider a nice tree decomposition $(T, \mathcal{X} = \{X_t\}_{t\in V(T)})$ of the graph of width at most $2c^2k^2$. Recall that for $t \in V(T)$, G_t is the graph induced on the vertices $\cup_{t'} X_{t'}$, where t' ranges over all descendants of t including t, and H_t is $G_t \setminus X_t$.

Let $P_{\theta_c}(G)$ denote the maximum number of vertex-disjoint θ_c minor models in G. Note that $P_{\theta_c}(G) \leq k'$. In our discussion, we abuse notation and use k' to denote $P_{\theta_c}(G)$. Consider the function $\mu : V(T) \rightarrow [k']$, defined as follows: $\mu(t) = P_{\theta_c}(H_t)$. The function μ is a good labeling function because:

- If t is a base node then $\mu(t) = 0$ as H_t is an empty graph.
- If t is an introduce node, then $\mu(t) = \mu(s)$, where s is the child of t. Indeed, this follows from the fact that the graphs H_t and H_s are exactly the same.
- If t is a join node, then $\mu(t) = \mu(s_1) + \mu(s_2)$, where s_1 and s_2 are the children of t. This follows from the fact that the bag X_t is a separator of G_t and $V(H_{s_1}) \cap V(H_{s_2}) = \emptyset$.
- If t is a forget node, then $\mu(t) \in \{\mu(s), \mu(s) + 1\}$, where s is the child of t. This is because H_t has at most one vertex more than H_s, which can add at most one to the number of vertex-disjoint θ_c minor models. the root r is ϕ, we have that $\mu(r) = k'$.

By definition, and by the convention that the bag X_r corresponding to the root r is ϕ we have that $\mu(r) = k'$. To find the desired θ_c-hitting set we give a recursive algorithm. We find a bag X in the given tree decomposition such that its removal allows us to decompose the graph into two parts such that there are no edges

Algorithm 1 Hit-Set(G)

Compute $(T, \mathcal{X} = \{X_t\}_{t \in V(T)})$, a nice tree decomposition of G. Now compute the function $\mu : V(T) \to [k']$, defined as follows: $\mu(t) = P_{\theta_c}(H_t)$.
if $(\mu(r) = 0)$ **then**
    Return $\emptyset$.
else
    Find the partitioning of the vertex set $V(G)$ into V_1, V_2 and X (a bag corresponding to a node in T) as described in Cases 1 and 2.
end if
Return $\left(X \bigcup \text{Hit-Set}(G[V_1]) \bigcup \text{Hit-Set}(G[V_2])\right)$.

from one part to another and the number of vertex-disjoint minor-models of θ_c in each part is essentially a constant fraction of the original. After this we find a hitting set in each of these graphs and then take the union of these sets, together with the bag we removed to get these graphs, to get the desired hitting set for the whole graph. Let $t \in V(T)$ be the node where $\mu(t) > 2k'/3$ and for each child t' of t, $\mu(t') \leq 2k'/3$. From the above observations, this node exists and is unique provided that $k' > 0$. Moreover, observe that t could either be a forget node or a join node. We distinguish these two cases.

- *Case 1.* If t is a forget node, we set $V_1 = V(H_{t'})$ and $V_2 = V(G) \setminus (V_1 \cup X_{t'})$ and observe that $P_{\theta_c}(G[V_i]) \leq \lfloor 2k'/3 \rfloor, i = 1, 2$. Also we set $X = X_{t'}$.
- *Case 2.* If t is a join node with children t_1 and t_2, we have that $\mu(t_i) \leq 2k'/3, i = 1, 2$. However, as $\mu(t_1) + \mu(t_2) > 2k'/3$, we also have that either $\mu(t_1) \geq k'/3$ or $\mu(t_2) \geq k'/3$. Without loss of generality we assume that $\mu(t_1) \geq k'/3$ and we set $V_1 = V(H_{t_1})$, $V_2 = V(G) \setminus (V_1 \cup X_{t_1})$ and $X = X_{t_1}$.

We present a detailed algorithm to find a hitting set in Algorithm 1. The algorithm Hit-Set(G) takes as input a graph G and returns a θ_c-hitting set for G. Now we bound the size of the hitting set returned by the algorithm. Let $\mathcal{S}(G, P_{\theta_c}(G)) \leq \mathcal{S}(G, k')$ be the size of the hitting set returned by Hit-Set(G). Then the value of $\mathcal{S}(G, P_{\theta_c}(G))$ is upper bounded by the following recurrence:

$$\mathcal{S}(G, k' = k-1) \leq \max_{1/3 \leq \alpha \leq 2/3} \left\{ \mathcal{S}(G[V_1], \alpha k') + \mathcal{S}(G[V_2], (1-\alpha)k') + 2c^2k^2 \right\}.$$

Note that $P_{\theta_c}(G[V_1]) + P_{\theta_c}(G[V_2]) \leq 2P_{\theta_c}(G)/3$. It is easy to see that the above recurrence solves to $O(k^2)$ using Akra-Bazzi Theorem [1]. This concludes the proof. □

3 Conclusion

In this short note we obtained a polynomial upper bound on the Erdős-Pósa property of a generalization of packing and covering cycles. An interesting question will be to classify those planar graphs H, such that $\mathcal{M}H$ has Erdős-Pósa property with a polynomial function on all graphs.

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